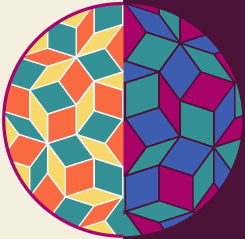
Finding classes of low complexity

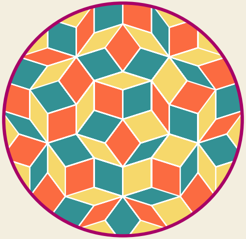


felix.reidl@gmail.com

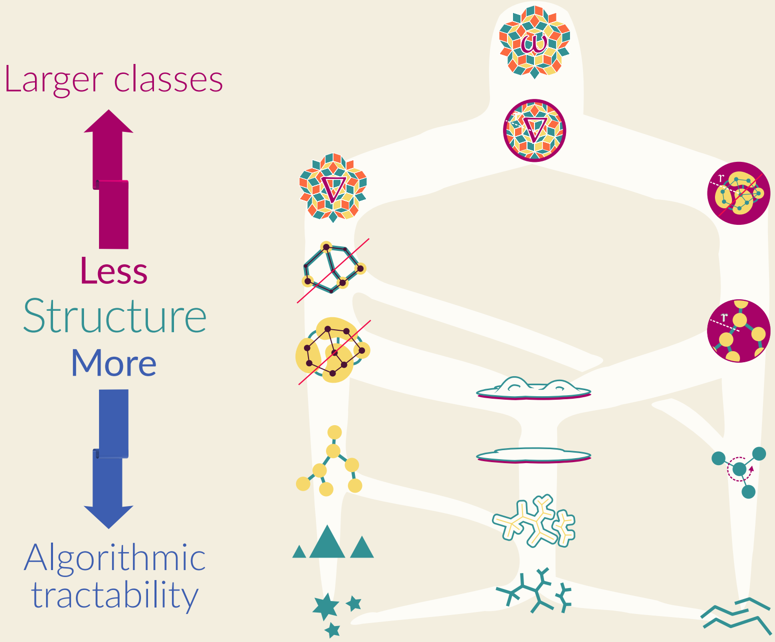
Workshop on structural sparsity, logic & algorithms.

Part I

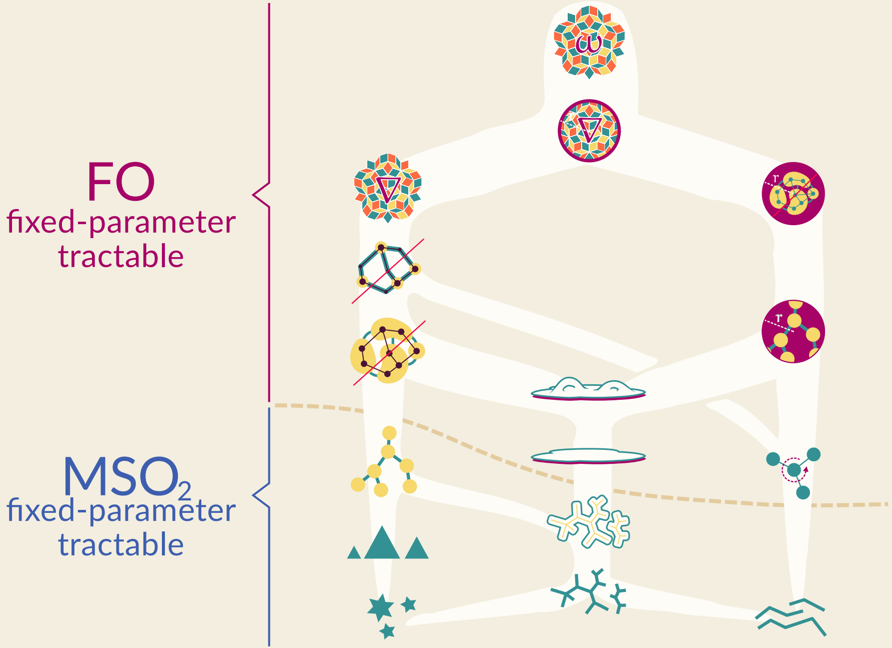
Sparse classes



The sparse class hierarchy



The sparse class hierarchy



Parameterised graph invariants

A **graph invariant** is an isomorphism invariant function that maps graphs to $\mathbb{R}^+$.

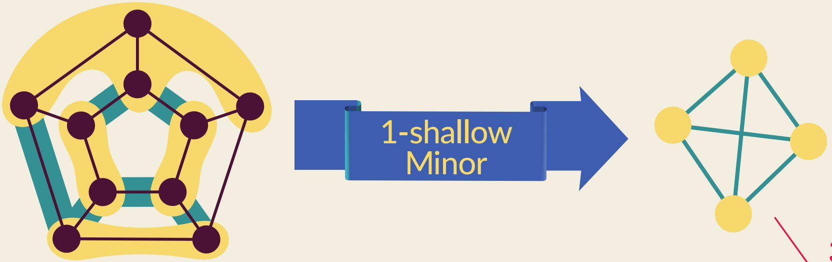
e.g. density, average degree, clique number, degeneracy treewidth, etc.

A **parameterised graph invariant** is a family of graph measures $(f_r)_{r\in\mathbb{N}_0}$.

A graph class $\mathcal{G}$ is f_r**-bounded** if there exists g s.t.

$$f_r(\mathcal{G}) = \limsup_{G\in\mathcal{G}} f_r(G) \leqslant g(r) \text{ for all } r.$$

Shallow minors & bounded expansion



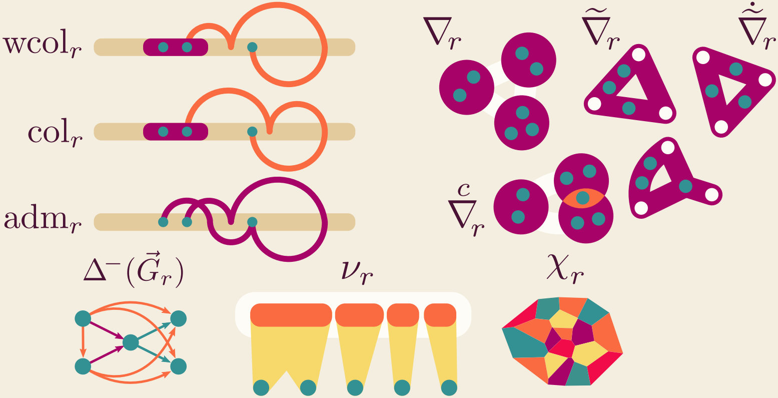
$$\frac{3}{2}$$

$$\nabla_r(G) = \max_{H \preccurlyeq_r G} \frac{|E(H)|}{|V(H)|}$$

A graph class has bounded expansion iff it is ∇_r-bounded.

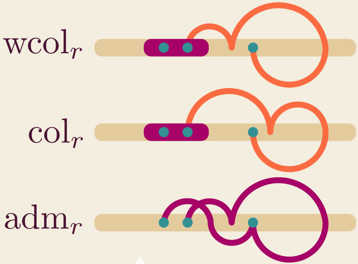
Bounded expansion

Nešetřil & Ossona de Mendez:
Many notions of f_r-boundedness are equivalent!



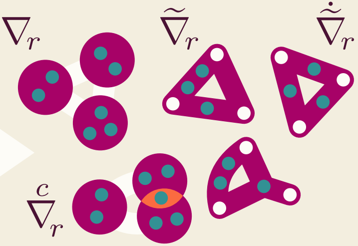
Nešetřil J, Ossona de Mendez P. **Sparsity**.
Algorithms and Combinatorics. 2012;28.

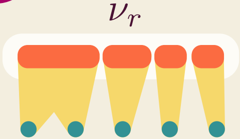
Bounded expansion



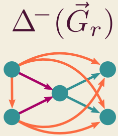
Size of r-*reachable sets* in ordering

Density of shallow minors



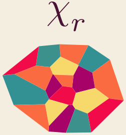


Normalized number of *traces* r-neighbourhoods leave in any subset



In-degree of r-step (d)tf-augmentation

Number of colours in r-treedepth colouring

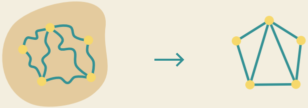


Nešetřil J, Ossona de Mendez P. **Sparsity**.
Algorithms and Combinatorics. 2012;28.

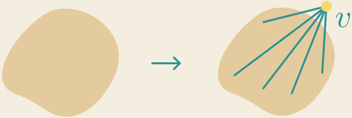
Bounded expansion is *robust*

Bounded expansion is preserved under the following class operations:

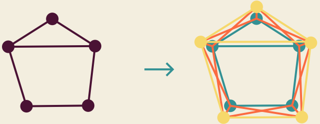
$\mathcal{G} \nabla r$ — Taking subgraphs / shallow minors



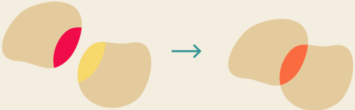
$\mathcal{G} + v$ — Adding an apex



$\mathcal{G} \bullet K_r$ — Lexproduct with a small graph

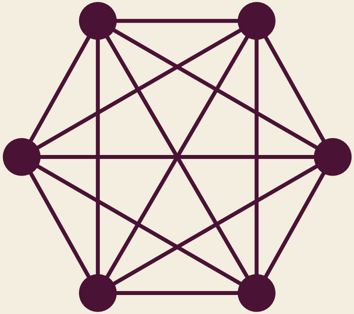


$\mathcal{G} \oplus_r \mathcal{H}$ — r-boundaried sums



But.

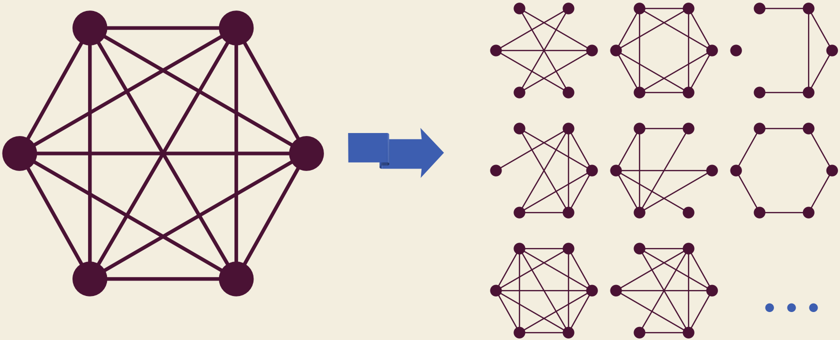
The humble clique



- Almost any problem is easy on cliques
- *Very* easy to remember (just recall n)
- Everyone is friends with everyone else! Nice!
- Absolutely not sparse

The humble clique

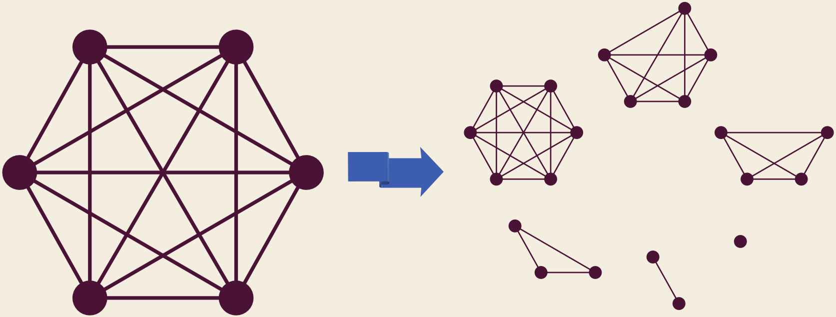
Why are cliques the ‘bad guys’ in sparse classes?



In monotone classes, cliques simply contain everything.

The humble clique

What if we restrict ourselves to induces subgraphs?



In *hereditary* classes, cliques are harmless!

Monotone classes: done.

FO model checking is FPT on nowhere dense classes.

Grohe M, Kreutzer S, Siebertz S.
Deciding first-order properties of nowhere dense graphs.
Journal of the ACM (JACM). 2017 Jun 16;64(3):17.

If $\mathcal{G}$ is somewhere dense and monotone, then the FO model checking problem on $\mathcal{G}$ is $AW[\star]$-complete.

Dawar A, Kreutzer S.
Parameterized complexity of first-order logic.
InElectronic Colloquium on Computational Complexity, TR09-131 2009 Dec 2 (p. 39).

Dvořák Z, Král D, Thomas R.
Testing first-order properties for subclasses of sparse graphs. Journal of the ACM (JACM). 2013 Oct 1;60(5):36.

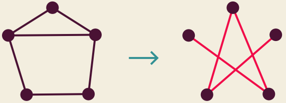
Stability = Nowhere Denseness in montone classes.

Adler H, Adler I.
Interpreting nowhere dense graph classes as a classical notion of model theory.
European Journal of Combinatorics. 2014 Feb 1;36:322-30.

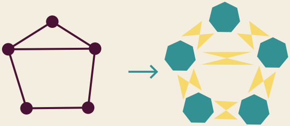
We want something *more* robust!

Bounded expansion is **not** preserved under the following class operations, but FO-tractability is:

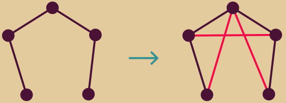
$\overline{\mathcal{G}}$ Complementation



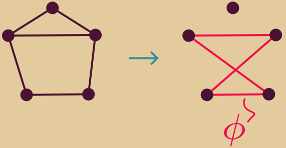
$\mathcal{G} \bullet K_{f(n)}$ Lexproduct with a **big** clique/stable set



$\mathcal{G}^r$ Taking powers

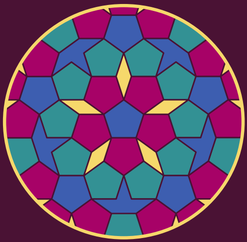


$\mathcal{I}_\phi(\mathcal{G})$ FO interpretations



Part II

Dense classes



The (maybe naive) goal

Find a notion of bounded complexity that

...generalizes bounded expansion
on hereditary classes

...is preserved under complementation,
set complements, lexproducts with simple
(but large) graphs

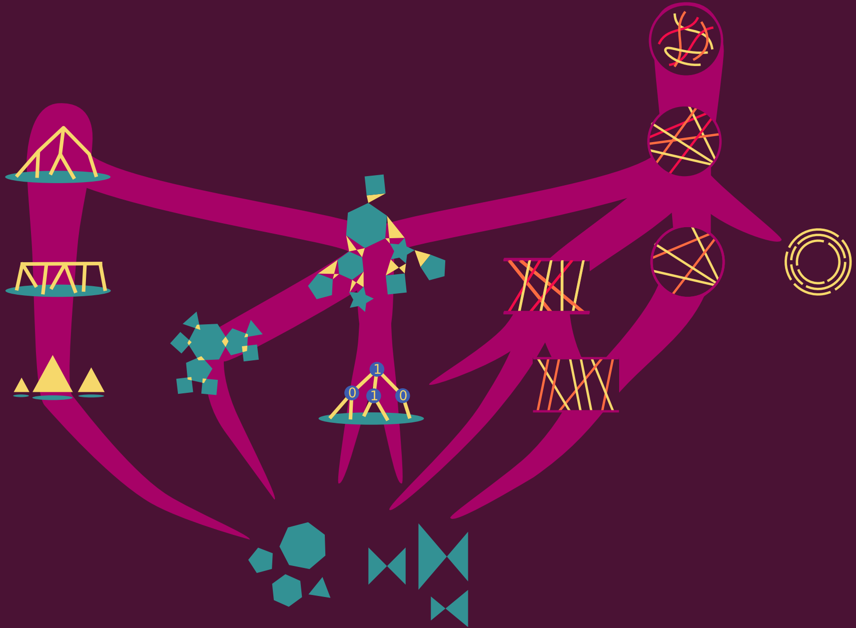
...is preserved under powers and
FO interpretations

...generalizes established & tractable
dense classes

...has a nowhere dense equivalent
(nowhere complex?)

...has nice algorithmic properties
(e.g. FO model checking in FPT time)

A selection of dense classes



The logical connection

Treedepth → MSO interpretation → Shrubdepth (SC-depth)

Treewidth → MSO interpretation → Rankwidth (Cliquewidth)

Bounded degree → FO interpretation → Near uniform

Gajarský J, Hliněný P, Obdržálek J, Lokshtanov D, Ramanujan MS. **A new perspective on FO model checking of dense graph classes.** InProceedings of the 31st Annual ACM/IEEE Symposium on Logic in Computer Science 2016 Jul 5 (pp. 176-184). ACM.

Bounded expansion → Power → Graphs with LSD

Kwon OJ, Pilipczuk M, Siebertz S. **On low rank-width colorings.** InInternational Workshop on Graph-Theoretic Concepts in Computer Science 2017 Jun 21 (pp. 372-385). Springer, Cham.

The logical connection

Treedepth → **MSO interpretation** → Shrubdepth (SC-depth)

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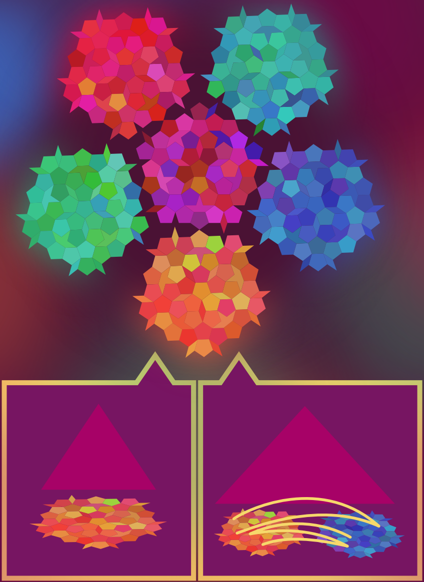
Bounded expansion → **FO interpretation** → Graphs with LSD

Gajarský J, Kreutzer S, Siebertz S, Toruńczyk S, Pilipczuk M, Ossona de Mendez P, Nešetřil J. **First-order interpretations of bounded expansion classes.** To appear.

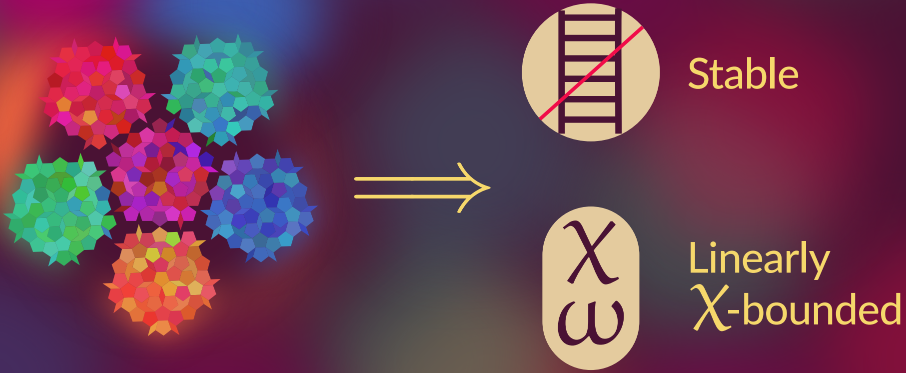
LSD: Low Shrubdepth Decomposition



Bounded shrubdepth

Bounded shrubdepth

LSD: Low Shrubdepth Decomposition



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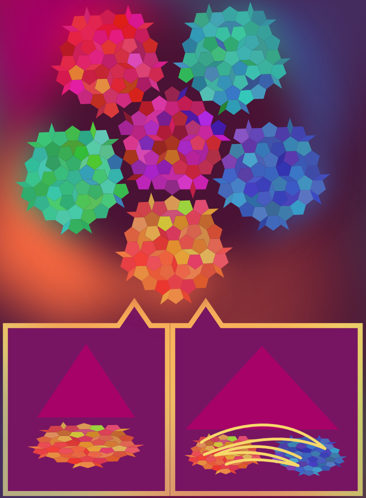

Bounded expansion

Structurally bounded expansion

Gajarský et al.:
A class has SBE iff it is an FO interpretation of a BE class.

Gajarský J, Kreutzer S, Siebertz S, Toruńczyk S, Pilipczuk M, Ossona de Mendez P, Nešetřil J. **First-order interpretations of bounded expansion classes.** To appear.

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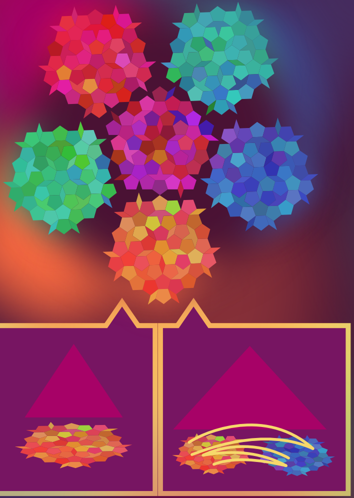

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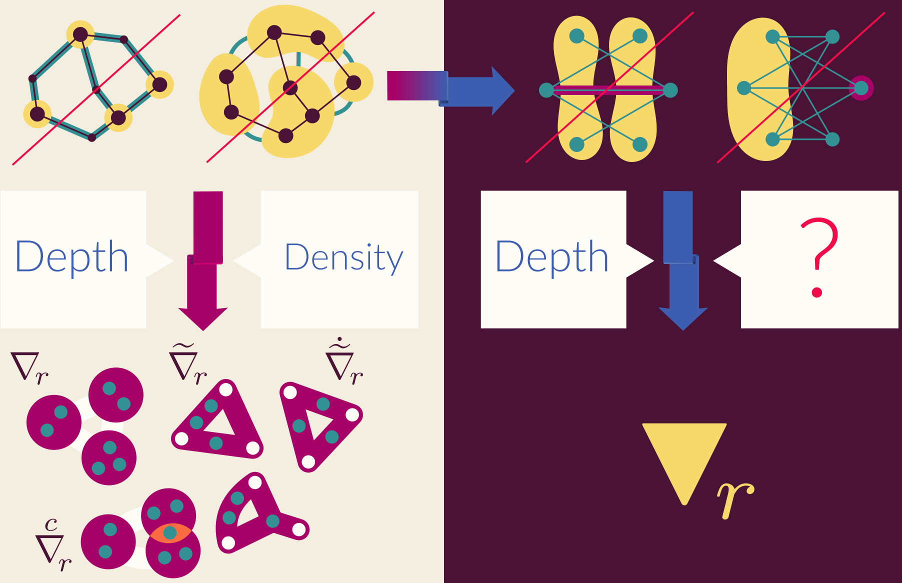
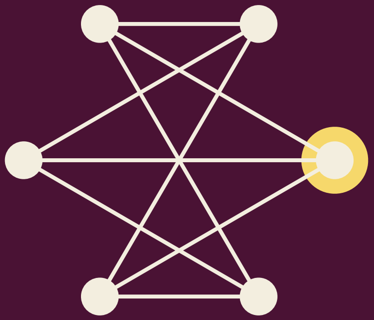
LSD is great, but is it what we need?

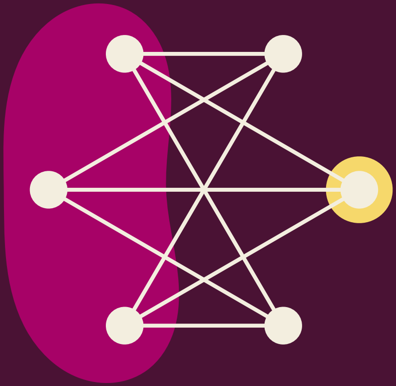
Forbidden (shallow) minors



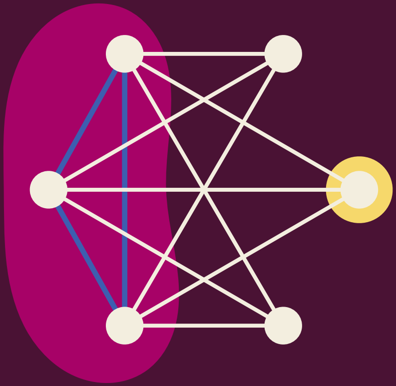
Vertex minors



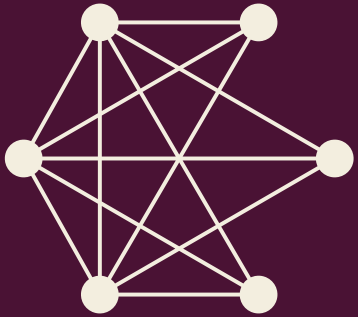
Vertex minors



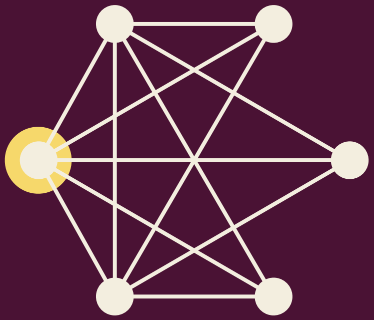
Vertex minors



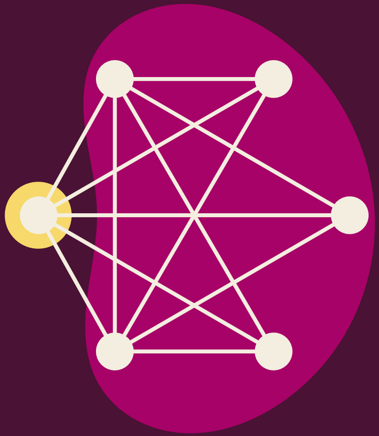
Vertex minors



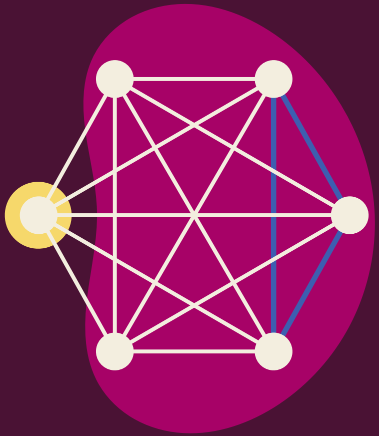
Vertex minors



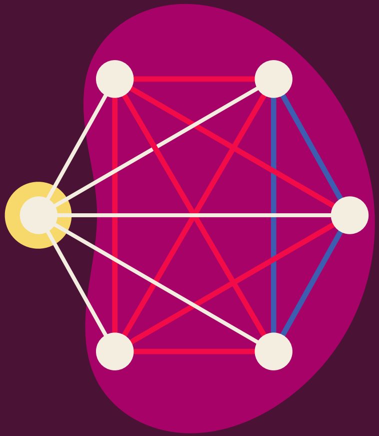
Vertex minors



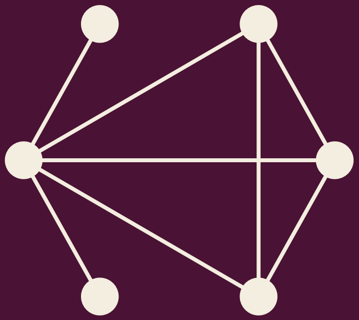
Vertex minors



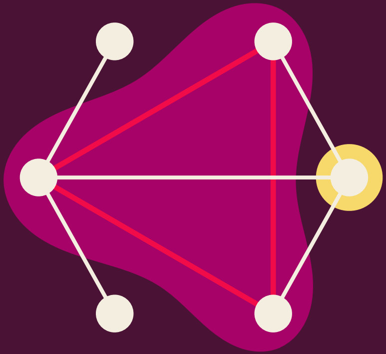
Vertex minors



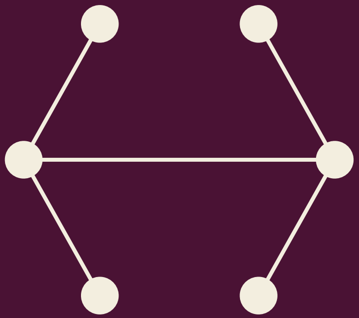
Vertex minors



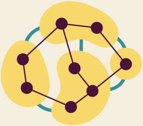
Vertex minors



Vertex minors



Minor vs v.minor



Induces WQO on graphs

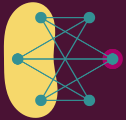
Robertson N, Seymour PD. **Graph minors. XX. Wagner's conjecture.** Journal of combinatorial theory, Series B. 2004 Nov 1;92(2):352-357.

Treewidth

Treedepth

Kwon OJ, Oum SI. **Graphs of small rank-width are pivot-minors of graphs of small tree-width.** Discrete Applied Mathematics. 2014 May 11;168:108-18.

Vertex minor



Induces WQO on graphs *of bounded rankwidth*

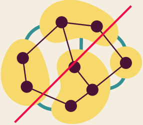
Oum SI. **Rank-width and well-quasi-ordering.** SIAM Journal on Discrete Mathematics. 2008 Mar 28;22(2):666-82.

Rankwidth
Cliquewidth

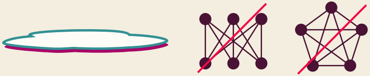
SC-depth
Shrubdepth

Hliněný P, Kwon OJ, Obdržálek J, Ordyniak S. **Tree-depth and vertex-minors.** European Journal of Combinatorics. 2016 Aug 1;56:46-56.

Excluded minor vs excluded v.minor



'Historical' ex.: planar graphs



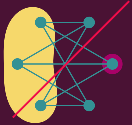
Kuratowski

Finite χ, degenerate

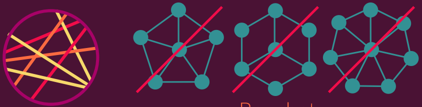
Mader W. **Homomorphieeigenschaften und mittlere Kantendichte von Graphen.** Mathematische Annalen. 1967 Dec 1;174(4):265-8.

Decomposition theorem

Robertson N, Seymour PD. **Graph minors. XVI. Excluding a non-planar graph**. Journal of Combinatorial Theory, Series B. 2003 Sep 1;89(1):43-76.



'Historical' ex.: circle graphs



Bouchet

χ-bounded (Geelen's conj.)?

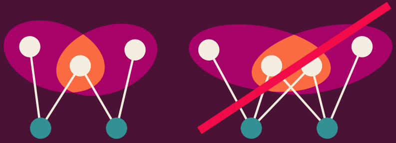
True for excluded wheels

Choi H, Kwon OJ, Oum SI, Wollan P. **Chi-boundedness of graph classes excluding wheel vertex-minors.** Electronic Notes in Discrete Mathematics. 2017 Aug 1;61:247-53.

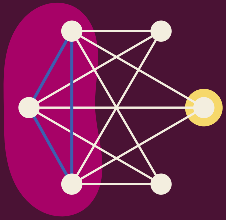
?

A notion of depth!

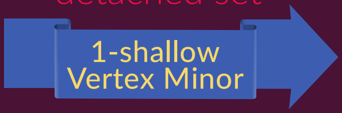
Detached: independent +
neighbourhoods intersect
in at most one vertex

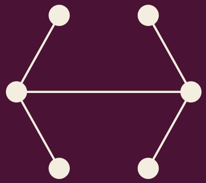


A notion of depth!

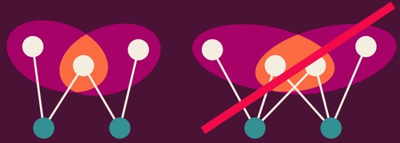


Locally complement a
detached set





Detached: independent + neighbourhoods intersect in at most one vertex



Using machinery by Gajarský et al.:

A graph class has SBE iff it can be constructed as shallow vertex minors from a BE class.

A notion of depth!

Using machinery by Gajarský et al.:

A graph class has SBE iff it can be constructed as shallow vertex minors from a BE class.

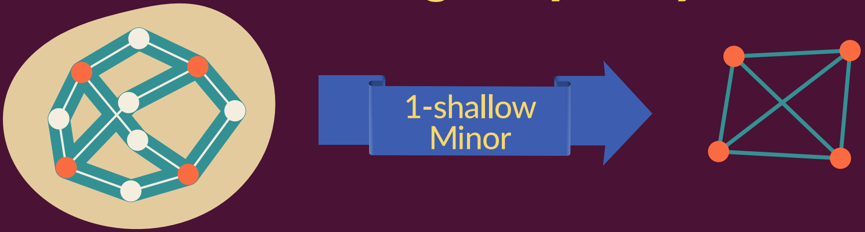
A notion of complexity?

Using machinery by Gajarský et al.:
A graph class has SBE iff it can be constructed as shallow vertex minors from a BE class.

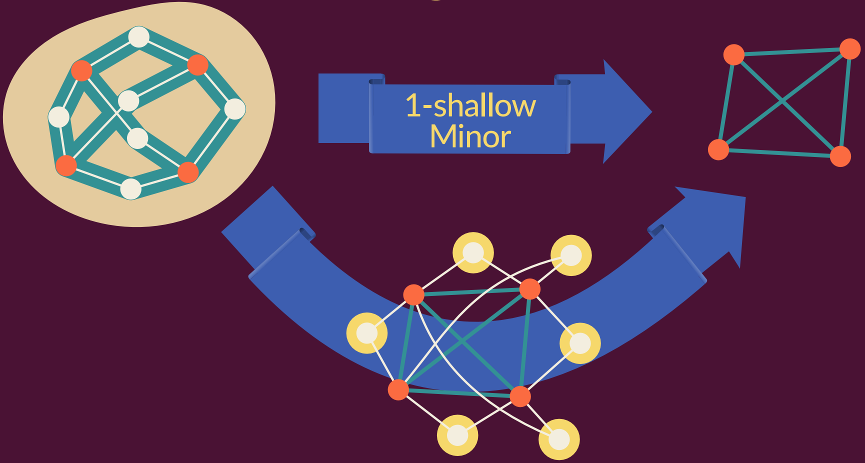
What we would like to have:
A graph class has SBE iff every r-shallow vertex minor has *complexity* bounded by f(r).

$$\blacktriangledown_r(G) = \max_{H \preccurlyeq_r^{\mathrm{vm}} G} \mathfrak{C}(H)$$

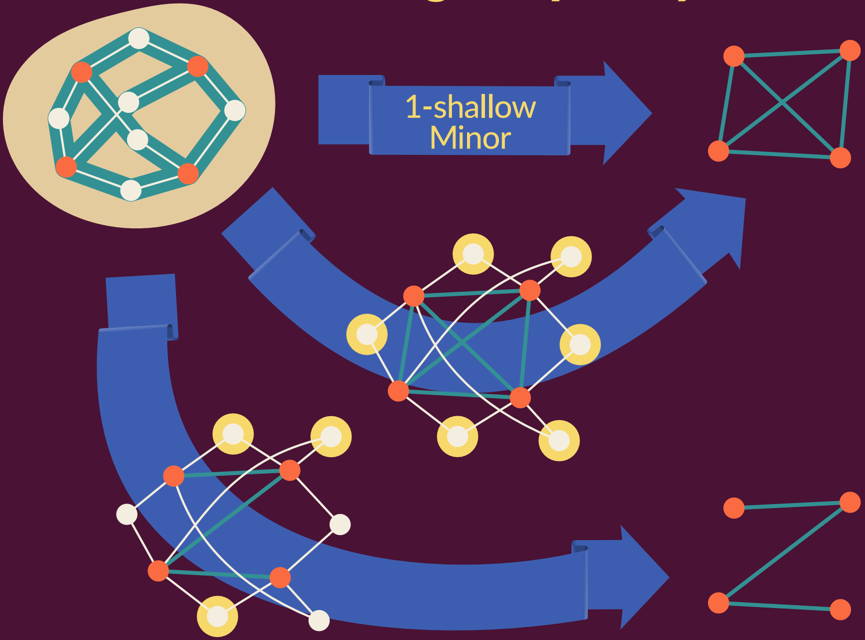
Avoiding *complexity*



Avoiding *complexity*

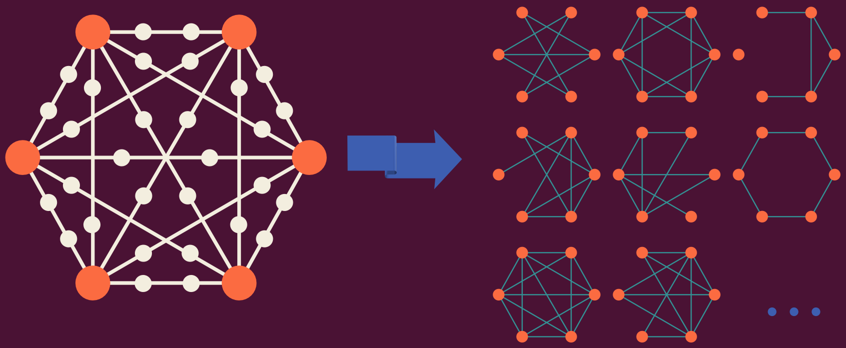


Avoiding *complexity*



Avoiding *complexity*

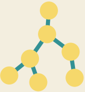
If a class contains arbitrarily large cliques as shallow induced subdivisions, then it contains *complex* graphs as shallow vertex minors.



To sparse, from low complexity

Treedepth

Treewidth



~~K_t~~ ~~$K_{t,t}$~~

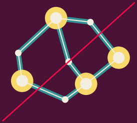
Shrubdepth
SC-depth

Rankwidth
Cliquewidth

Fomin FV, Oum SI, Thilikos DM. **Rank-width and tree-width of H-minor-free graphs.** European Journal of Combinatorics. 2010 Oct 1;31(7):1617-28.

Bounded expansion

~~K_t~~ ~~$K_{t,t}$~~

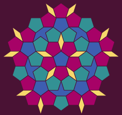


No dense induced subdivision

Dvořák Z. **Induced subdivisions and bounded expansion.** arXiv preprint arXiv:1706.05766. 2017 Jun 19

Bounded expansion

~~K_t~~ ~~$K_{t,t}$~~

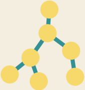


Structurally bounded expansion

Gajarský J, Kreutzer S, Siebertz S, Toruńczyk S, Pilipczuk M, Ossona de Mendez P, Nešetřil J. **First-order interpretations of bounded expansion classes.** To appear.

To sparse, from low complexity

Treedepth 

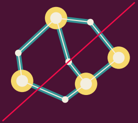
Treewidth 

Shrubdepth
SC-depth

Rankwidth
Cliquewidth

Fomin FV, Oum SI, Thilikos DM. **Rank-width and tree-width of H-minor-free graphs.** European Journal of Combinatorics. 2010 Oct 1;31(7):1617-28.

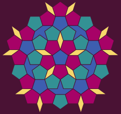
Bounded expansion 



No dense induced subdivision

Dvořák Z.
Induced subdivisions and bounded expansion.
arXiv preprint arXiv:1706.05766. 2017 Jun 19

Nowhere dense 



Structurally nowhere dense

R. F, Siebertz S, Sullivan B, Ossona de Mendez P, Nešetřil J. **WIP**

Part III

Summary

From sparse to low complexity

Treedepth → FO interpretation → Shrubdepth (SC-depth)

Bounded degree → FO interpretation → Near uniform

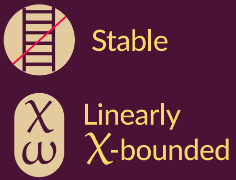
Bounded expansion → subset complement → Obfuscated bounded expansion

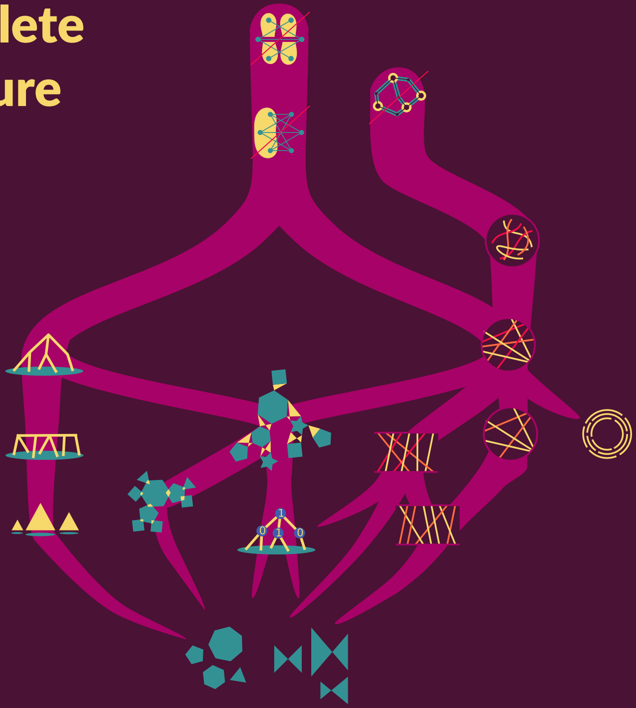
Gajarsky J, Kral D. **Deobfuscating sparse graphs.** arXiv preprint arXiv:1709.09985. 2017 Sep 28.

$\cap$

Bounded expansion → FO interpretation → Structurally bounded expansion

An incomplete dense picture





Circle graphs—not the good guys?

Linearly χ-bounded ?

Stable ?

Circle graphs—not the good guys?

Linearly χ-bounded !

Stable ?

Kostochka: There exist circle graphs s.t.

$$\chi = \Omega(\omega \log \omega)$$

A. Kostochka. **On upper bounds on chromatic numbers of graphs.** Transaction of the Institute of mathematics, 10:204–226, 1988.

Bipartite circle graphs—not the good guys?

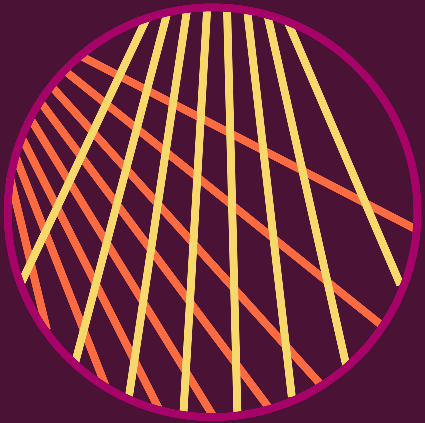
Linearly χ-bounded !

Stable ?

Bipartite circle graphs—not the good guys?

Linearly χ-bounded !

Stable !



Good guy circle graph???

Linearly
χ-bounded

Stable

Good guy circle graph???

Linearly χ-bounded

Stable

Good guy circle graph???

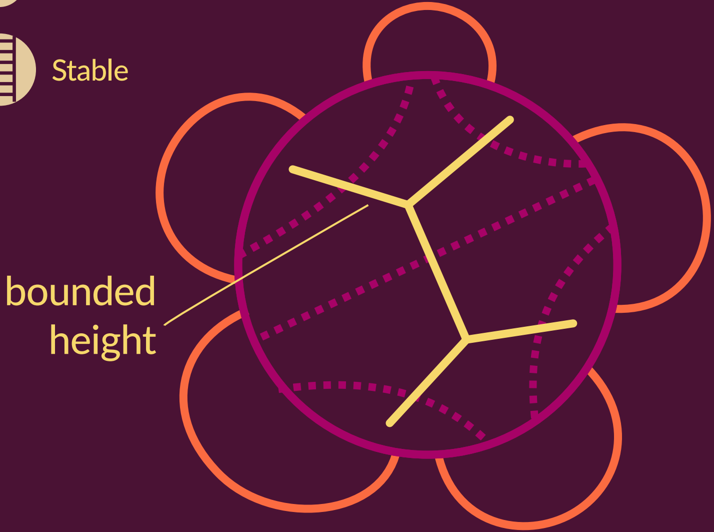
Linearly χ-bounded

Stable

Good guy circle graph???

Linearly
χ-bounded

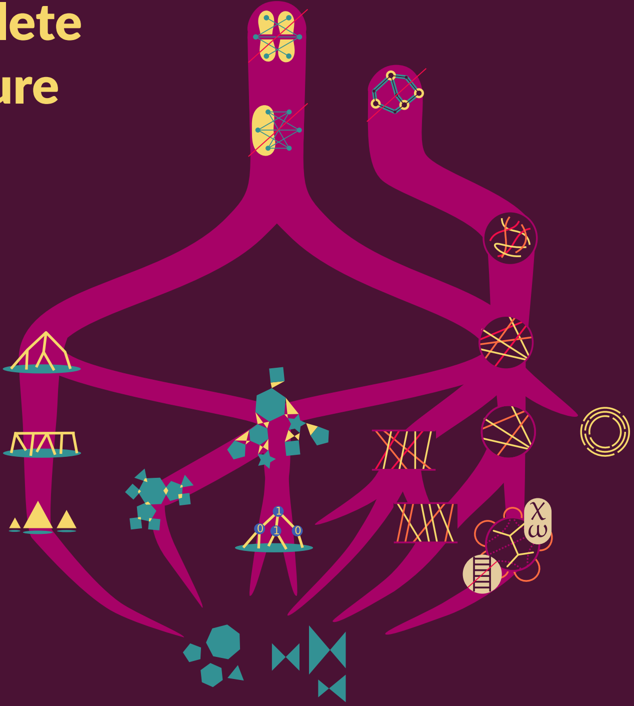
Stable



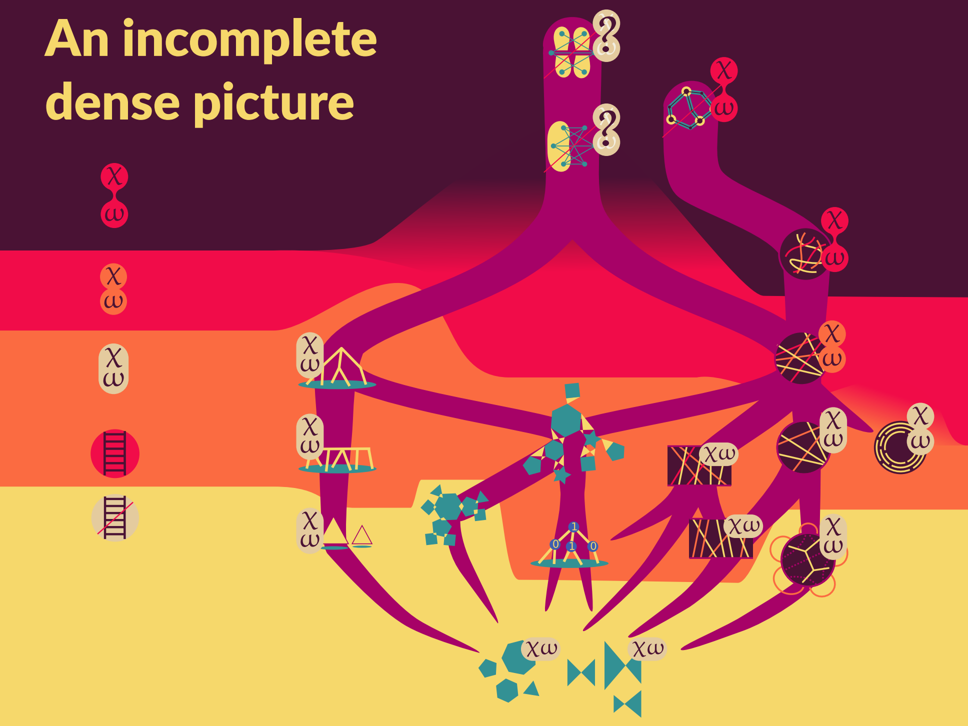
An incomplete dense picture

Stable

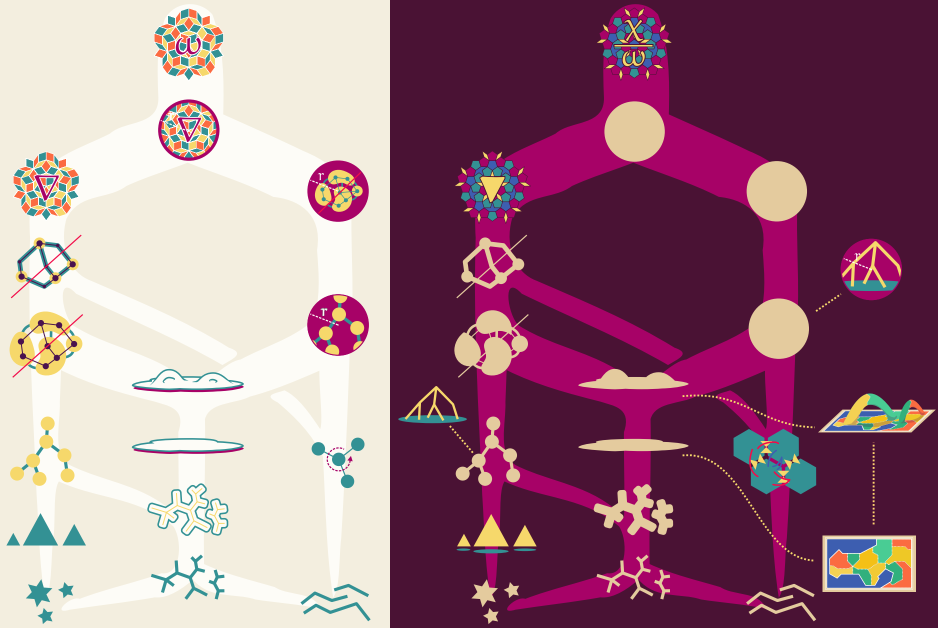
Linearly χ-bounded



An incomplete dense picture



Interpreted hierarchy



THANKS!

Questions?