Structural Sparseness & Complex Networks

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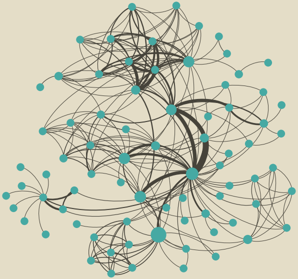
Joint work with Erik Demaine, Peter Rossmanith, Fernando Sánchez Villaamil, Somnath Sikdar, and Blair D. Sullivan

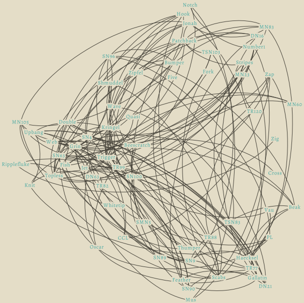
SIAM NS '16

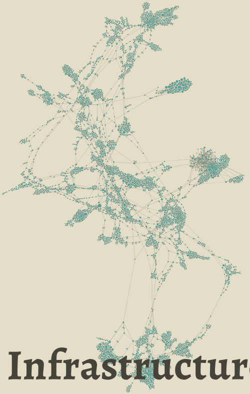
Complex Networks

= Real world graphs

(+ a lot of annotations)

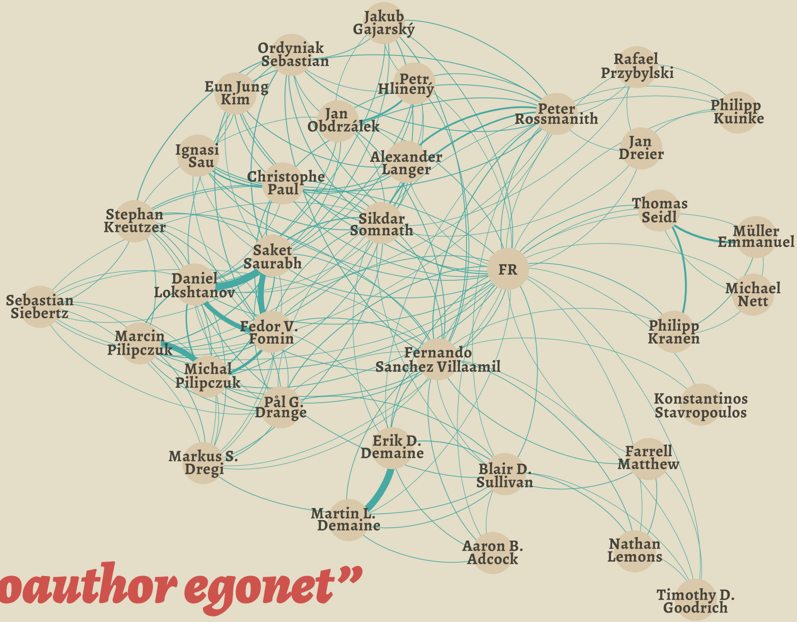






Sociology	Biology	Infrastructure
Friendships,	Gene-gene interactions,	Road networks,
Collaborations,	Protein-protein inter.,	Power grids,
Communication,	Neural networks,	Computer networks,
...	...	....

Speaking of networks



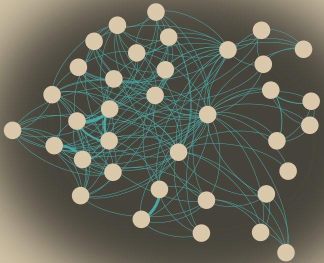
“Coauthor egonet”

Properties of Real Graphs

Giant connected component

Maximum degree around $n^{\varepsilon},\ \varepsilon < 1$

Clustered



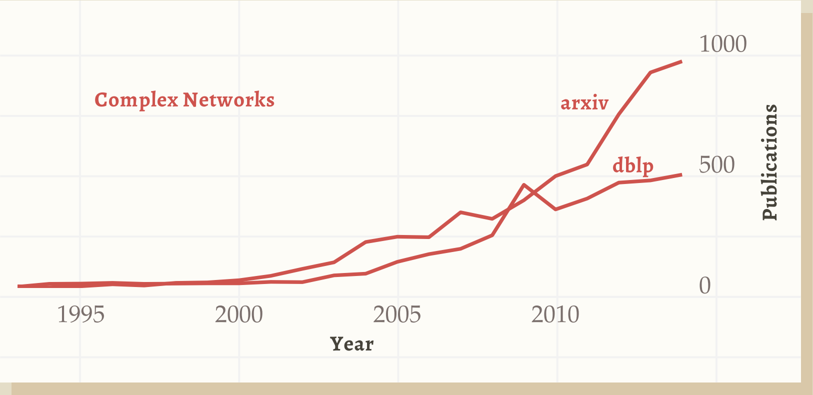
Low diameter

Random (but not uniform)

Sparse

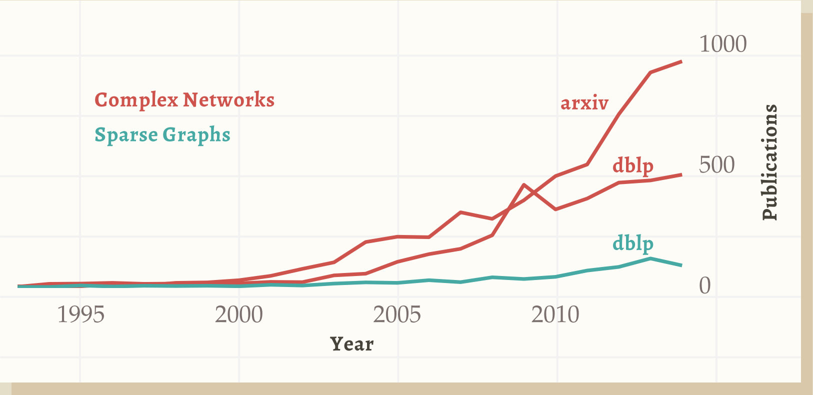
A Booming Field

1) We collect a *lot* of network data
2) We need to compute things on them
3) Sparse graphs admit nice algorithms



A Booming Field

1) We collect a *lot* of **network data**
2) We need to **compute** things on them
3) Sparse graphs admit **nice algorithms**



Sparse Hierarchy

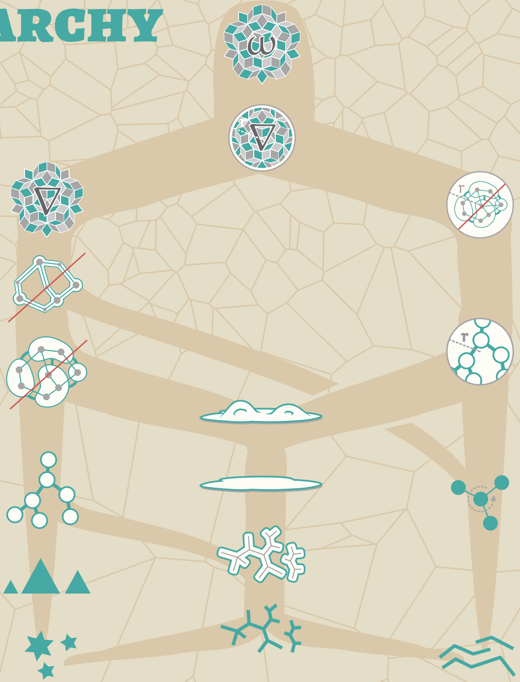
Applicability to real world

Less

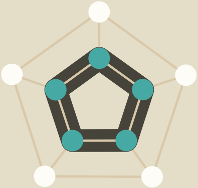
Structure

More

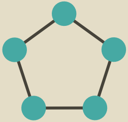
Algorithmic tractability

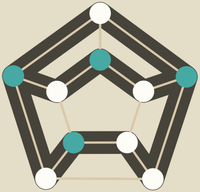


Substructures

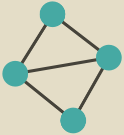


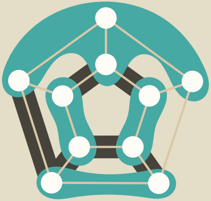
Select vertices, connect by edges



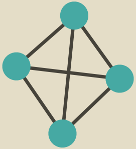


Select vertices, connect by vertex-disjoint paths





Select connected, disjoint subgraphs, connect by edges



Forbidden Substructures

H does not appear as a subgraph.

H does not appear as a topological minor.

H does not appear as a minor.

Forbidden Substructures

 does not appear as a subgraph.

= Triangle-free graphs

 does not appear as a topological minor.

= Forests

 does not appear as a minor.

= Forests

Forbidden Substructures

H does not appear as a topological minor.

H does not appear as a minor.

Forbidden Substructures

Every top. minor is sparse.

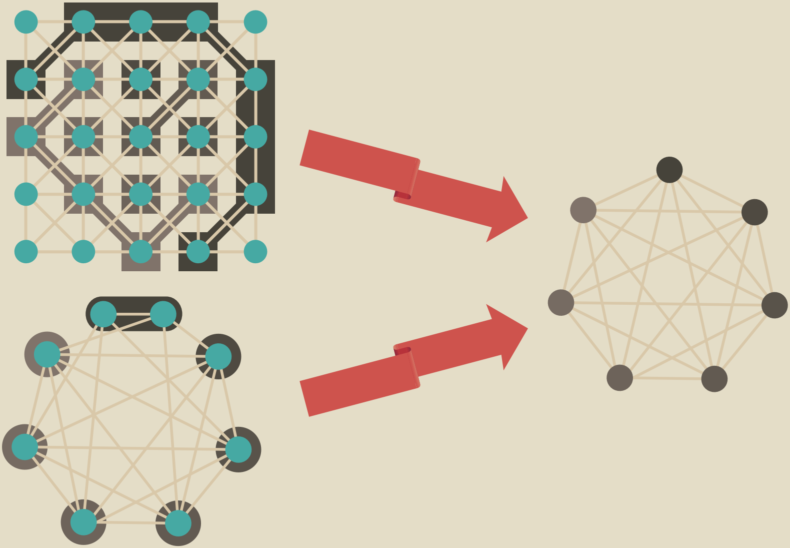
Every minor is sparse.

Forbidden Substructures

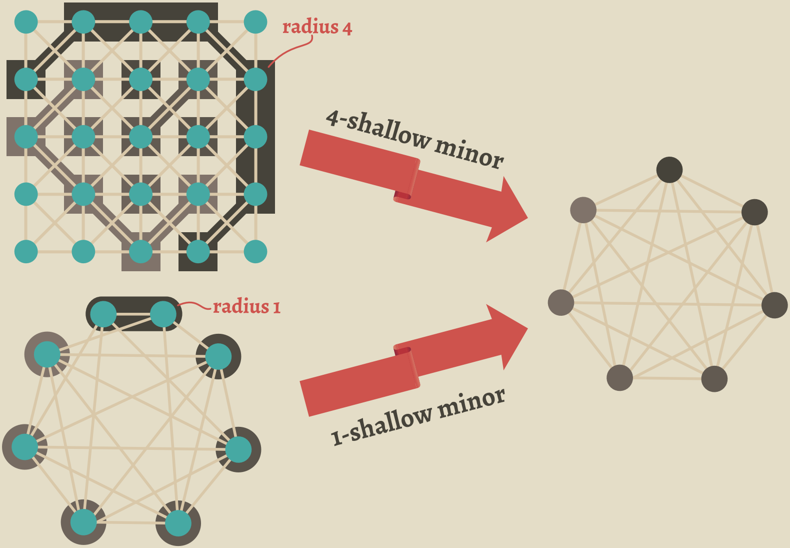
Every top. minor is sparse.
= Class exludes K_t as a top. minor.

Every minor is sparse.
= Class exludes K_t as a minor.

Not all minors are equal!



NOT ALL MINORS ARE EQUAL!



Bounded expansion

A graph class has *bounded expansion* iff there exists a function f such that every r-shallow minor has density at most $f(r)$.

"We allow dense minors in our graph, but only on a large scale"

Bounded expansion

A graph class has *bounded expansion* iff there exists a function f such that every r-shallow minor has density at most $f(r)$.

"We allow dense minors in our graph, but only on a large scale"

A graph class has *bounded expansion* iff there exists a function f such that every r-shallow topological minor has density at most $f(r)$.

Sparse Hierarchy

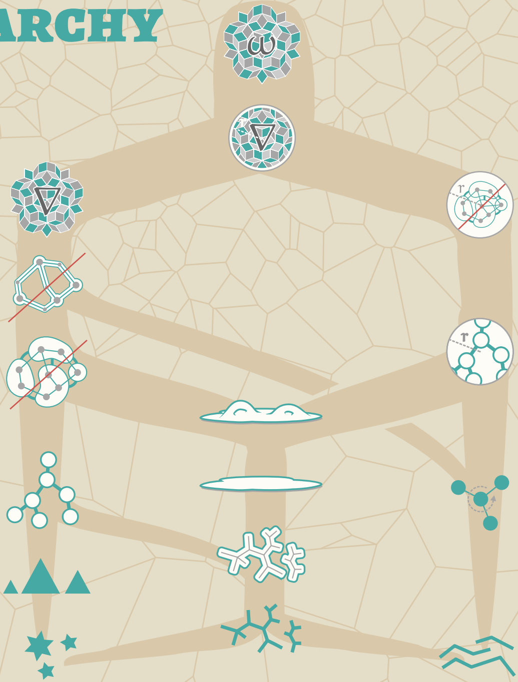
Applicability
to real world

Less

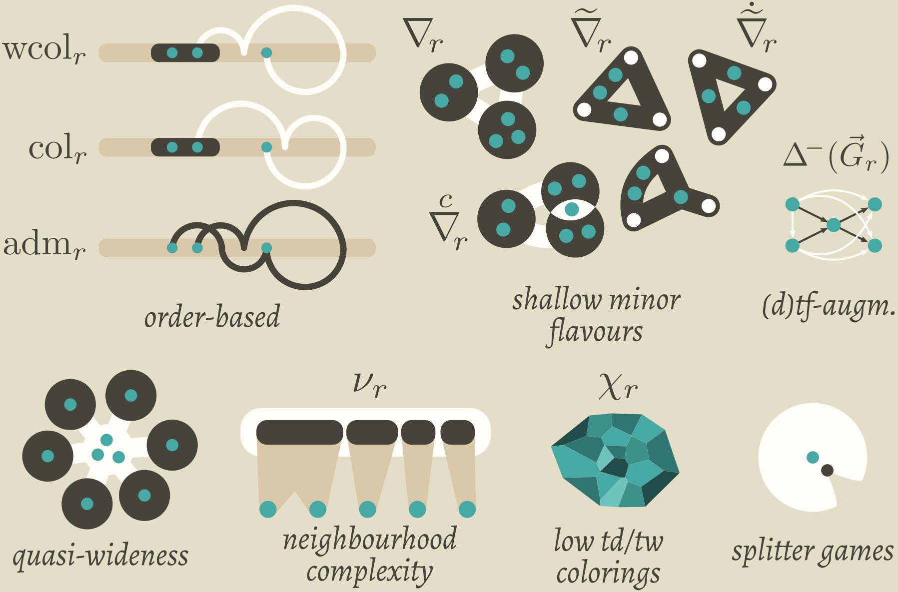
Structure

More

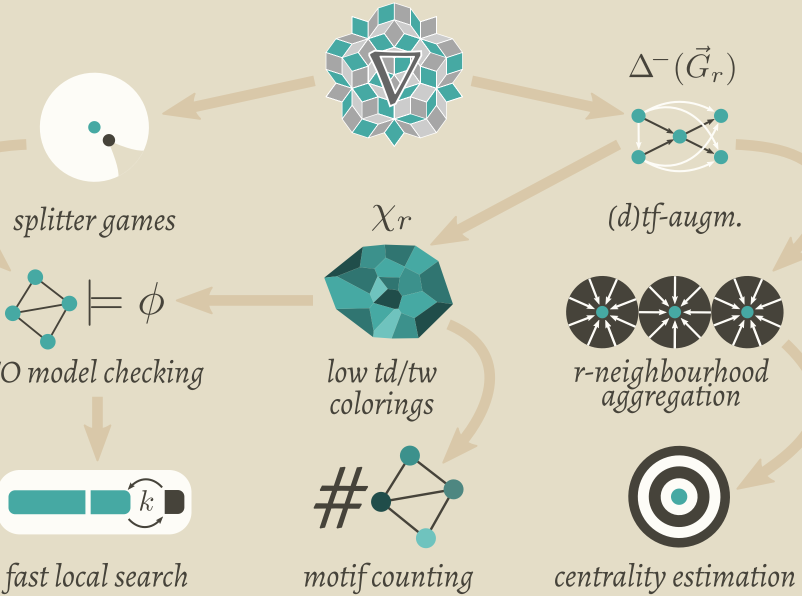
Algorithmic
tractability



Choose your poison



Applications

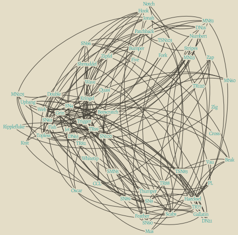


From data to theory

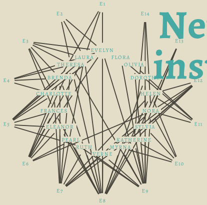
$$Pr[\|G\| \geq \xi k] \leq \left(\frac{e\beta D^2}{2n\xi k e^{D^2/2n}}\right)^{\xi k}$$

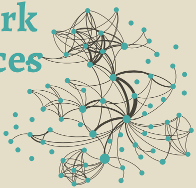
Mathematical theory

???



Network instances





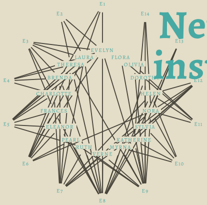
From data to theory

$$Pr[\|G\| \geq \xi k] \leq \left(\frac{e\beta D^2}{2n\xi k e^{D^2/2n}}\right)^{\xi k}$$

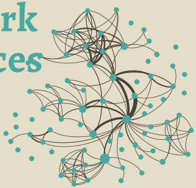
Mathematical theory

Network model

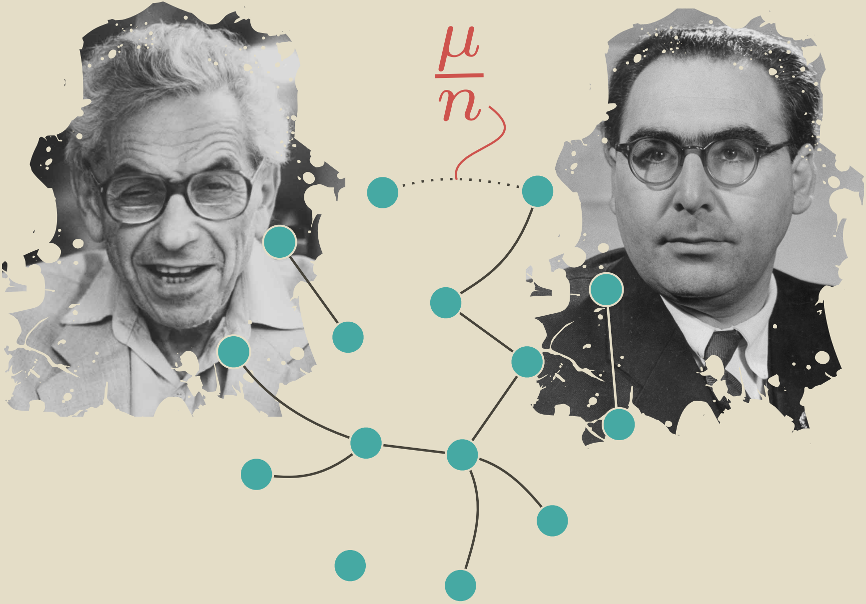
- Random network
- Tunable parameters
- Replicates some statistics



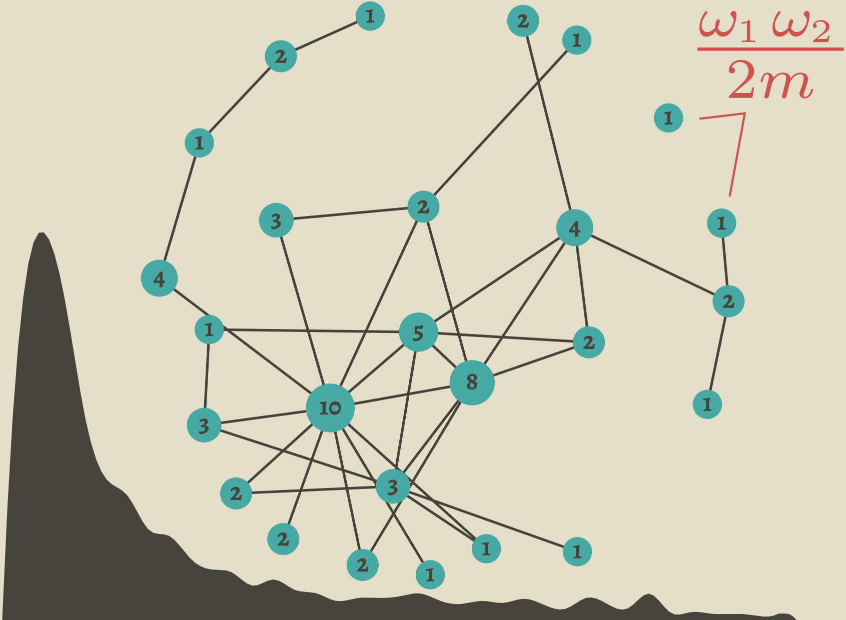
Network instances



Erdős–Rényi: Structurally Sparse

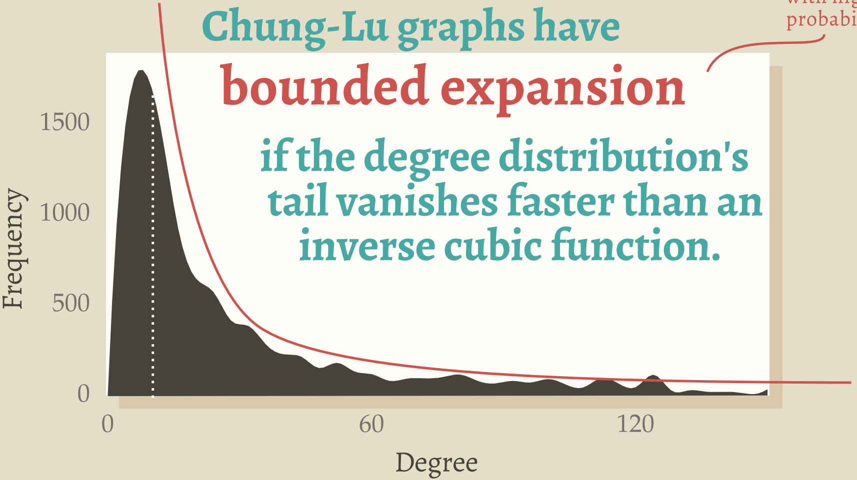


Chung–Lu: Better by a degree



The devil is in the D-tail

Chung-Lu graphs have **bounded expansion** (with high probability) if the degree distribution's tail vanishes faster than an inverse cubic function.



(Proof idea: couple occurences of shallow top. minors to subgraphs in a different Chung-Lu graph, bound probability of dense subgraph in that graph.)

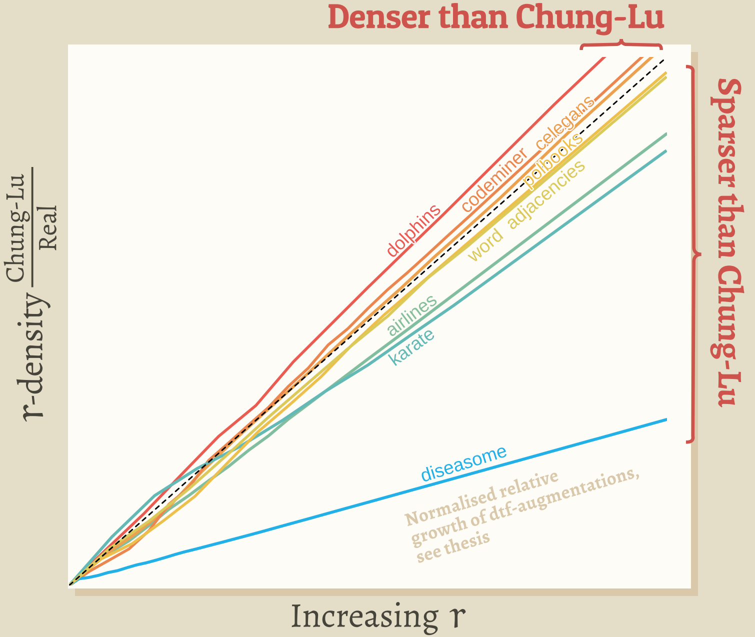
Structural phase transition

- **Degree distribution with tail-bound $\frac{1}{h(d)}$:**

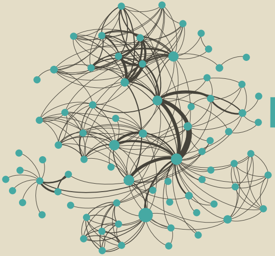
$$h(d) = \begin{cases} \Omega(d^{3+\epsilon}) & \text{bounded expansion} \\ \Theta(d^{3+o(1)}) & \text{nowhere dense} \\ O(d^{3-\epsilon}) & \text{somewhere dense} \end{cases}$$

- **Proof idea for lower bounds: more coupling**
- **The same works for the so-called 'configuration model'**
- **Also works for similar models with non-vanishing clustering**

Comparative structural density



Real Structural Sparseness



Statistical test of degree distribution

h(d) looks subcubic → Cannot decide

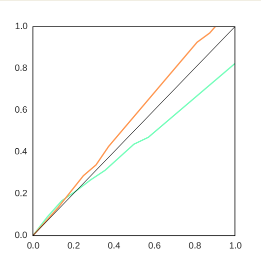
h(d) looks supercubic ↓

Compare density to Chung-Lu w/ same degree distribution

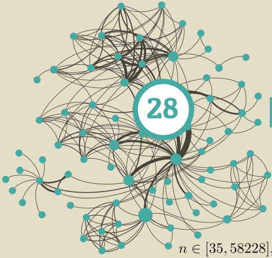
Denser than Chung-Lu → Cannot decide

Sparser than Chung-Lu ↓

Structurally sparse!



Real Structural Sparseness



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$n \in [35, 58228]$,
$m \in [78, 214078]$

Statistical test of degree distribution

h(d) looks subcubic → **6** Cannot decide

h(d) looks supercubic ↓

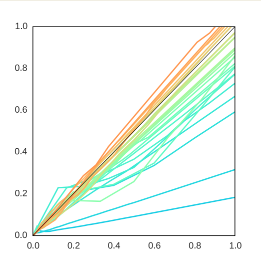
22 Compare density to Chung-Lu w/ same degree distribution

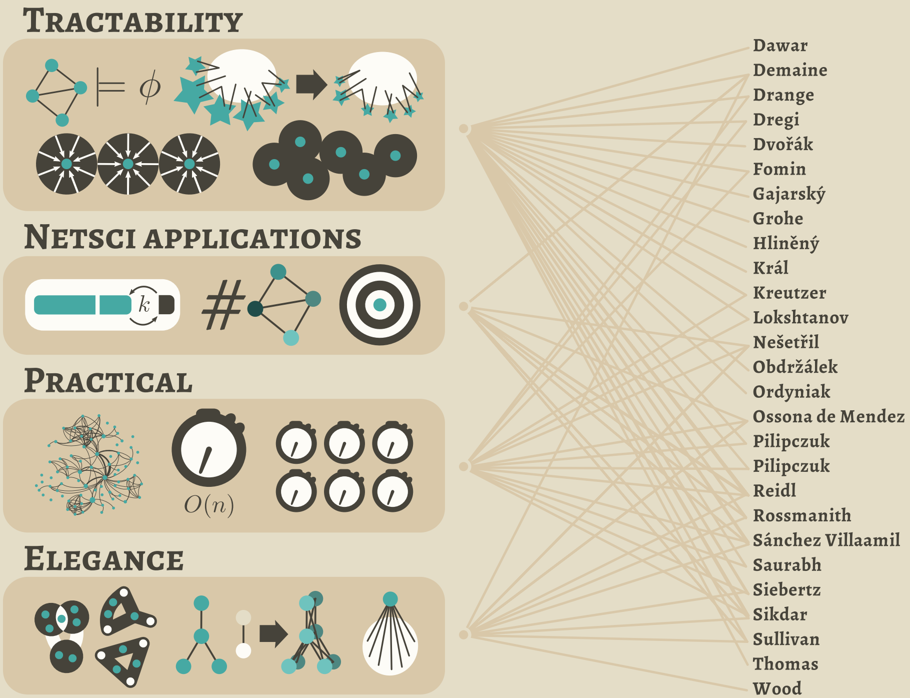
Denser than Chung-Lu → **8** Cannot decide

7 of those are probably sparse

Sparser than Chung-Lu ↓

14 Structurally sparse!




Tractability
$\models \phi$
Netsci applications
k
#
Practical
$O(n)$
Elegance
Dawar
Demaine
Drange
Dregi
Dvořák
Fomin
Gajarský
Grohe
Hliněný
Král
Kreutzer
Lokshtanov
Nešetřil
Obdržálek
Ordyniak
Ossona de Mendez
Pilipczuk
Pilipczuk
Reidl
Rossmanith
Sánchez Villaamil
Saurabh
Siebertz
Sikdar
Sullivan
Thomas
Wood

THANKS!

Questions?

References

Reidl, F. (2015). **Structural sparseness and complex networks.**

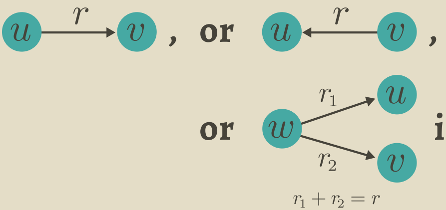
Demaine, E. D., Reidl, F., Rossmanith, P., Villaamil, F. S., Sikdar, S., & Sullivan, B. D. (2014). **Structural sparsity of complex networks: Bounded expansion in random models and real-world graphs.** arXiv preprint arXiv:1406.2587.

DTF-AUGMENTATIONS

Theorem: Let $G \in \mathcal{G}$ from a bounded expansion class. There exists a sequence $\vec{G}_1, \vec{G}_2, \ldots$ of edge-weighted digraphs such that

a) $\Delta^-(\vec{G}_r) \leqslant f(r)$

b) For all u, v with $\mathrm{dist}_G(u,v) \leqslant r$ either

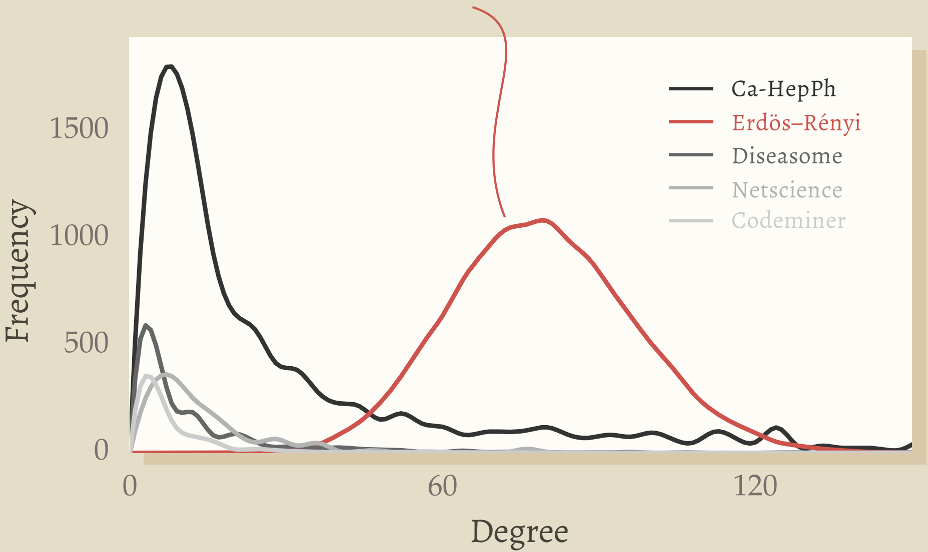


is present in $\vec{G}_r$.

Moreover, for fixed r this sequence is computable in linear time.

Deficiency of ER

Unrealistic degree distribution



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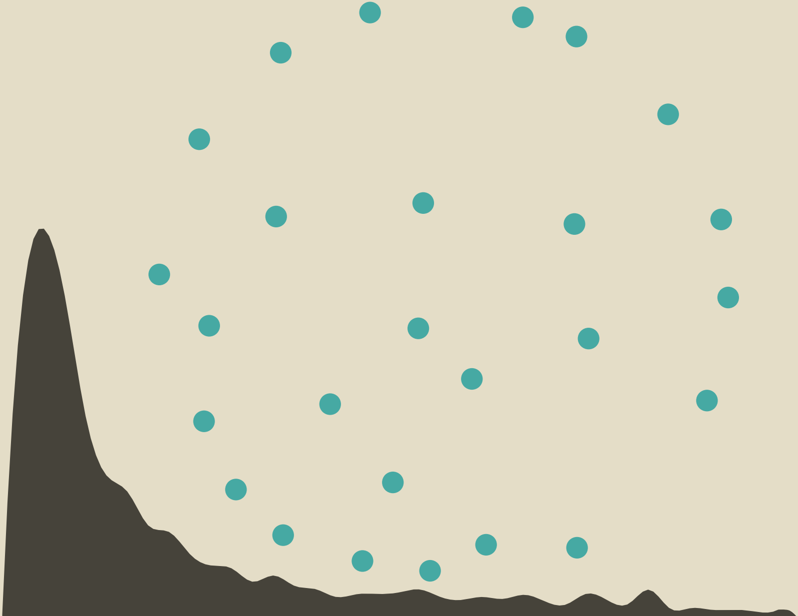

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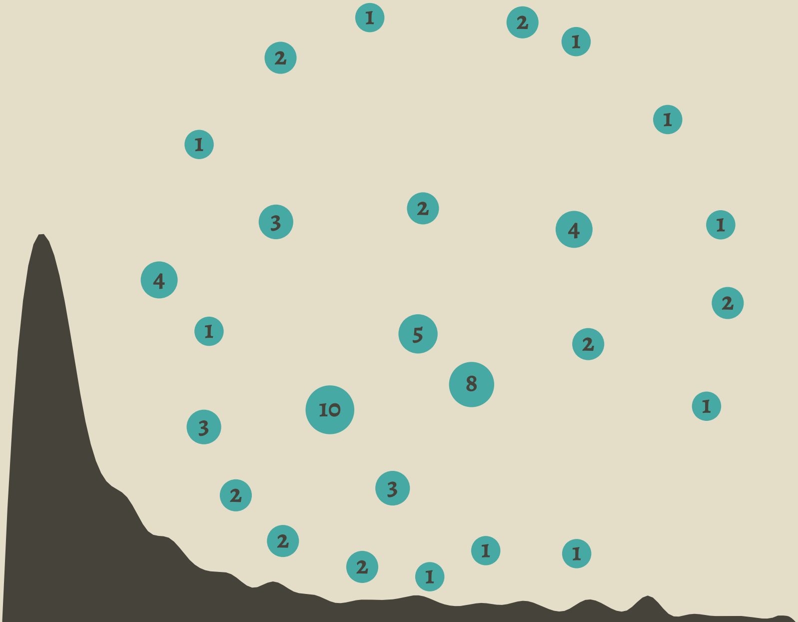

Chung–Lu: Better by a degree