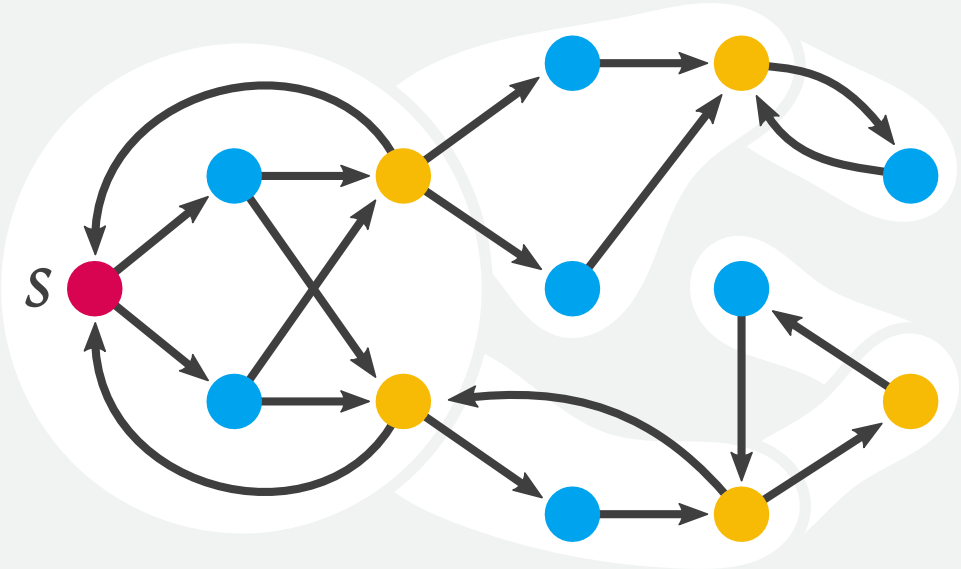
k-Distinct In- and Out-Branchings

Gregory Gutin[1] **Felix Reidl**[1,2] Magnus Wahlström[1]



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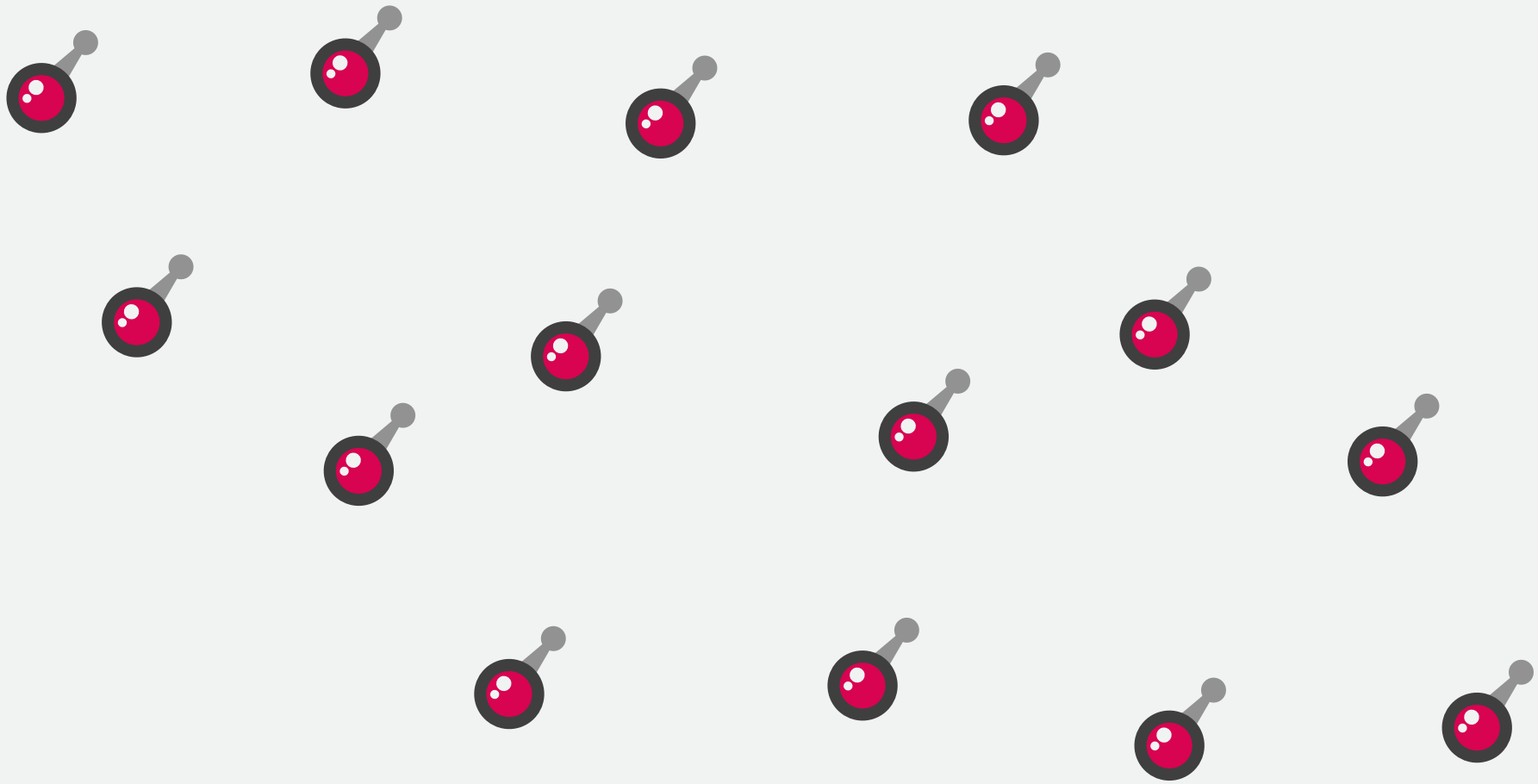
State University, USA

Routing in Sensor Networks



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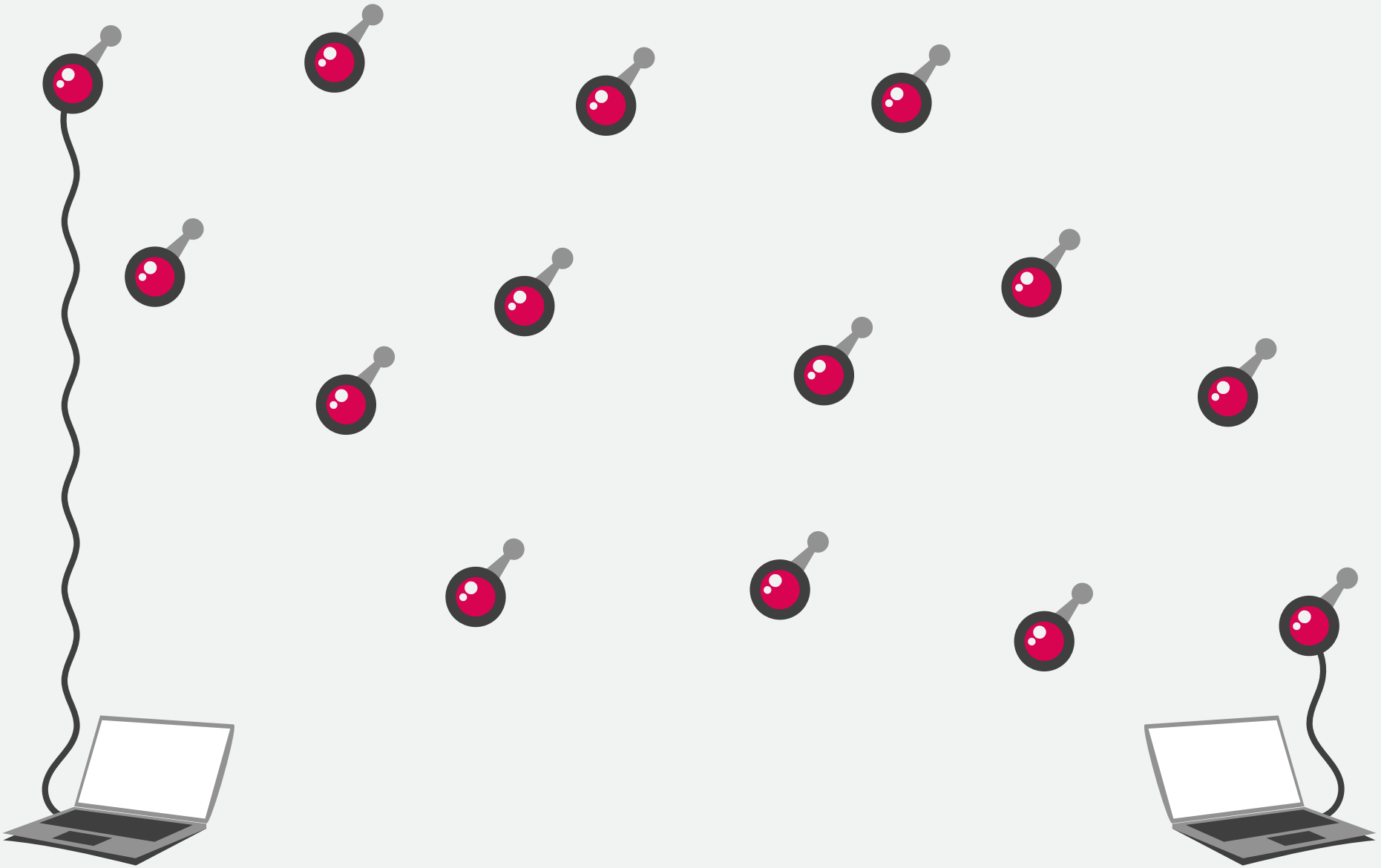

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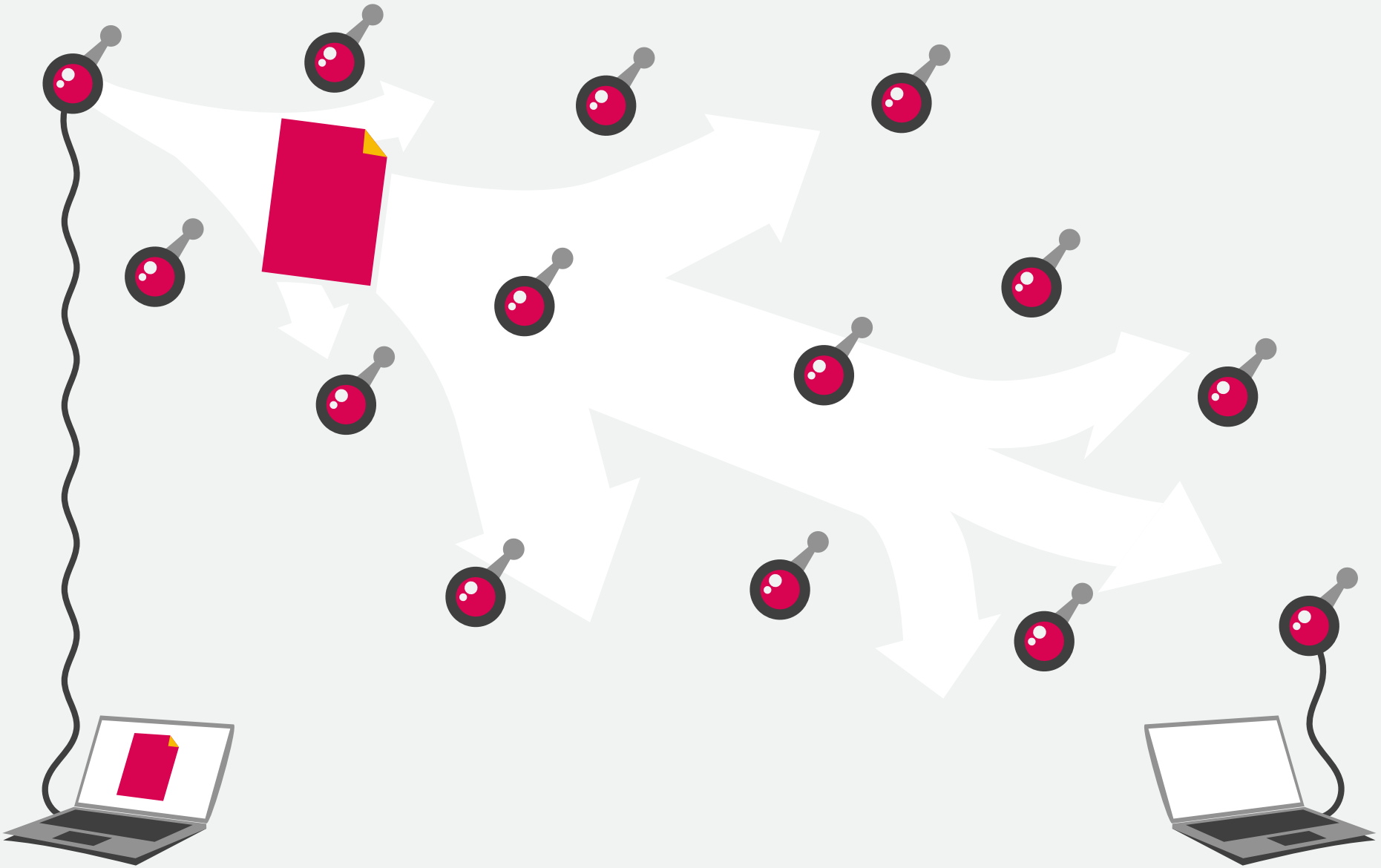

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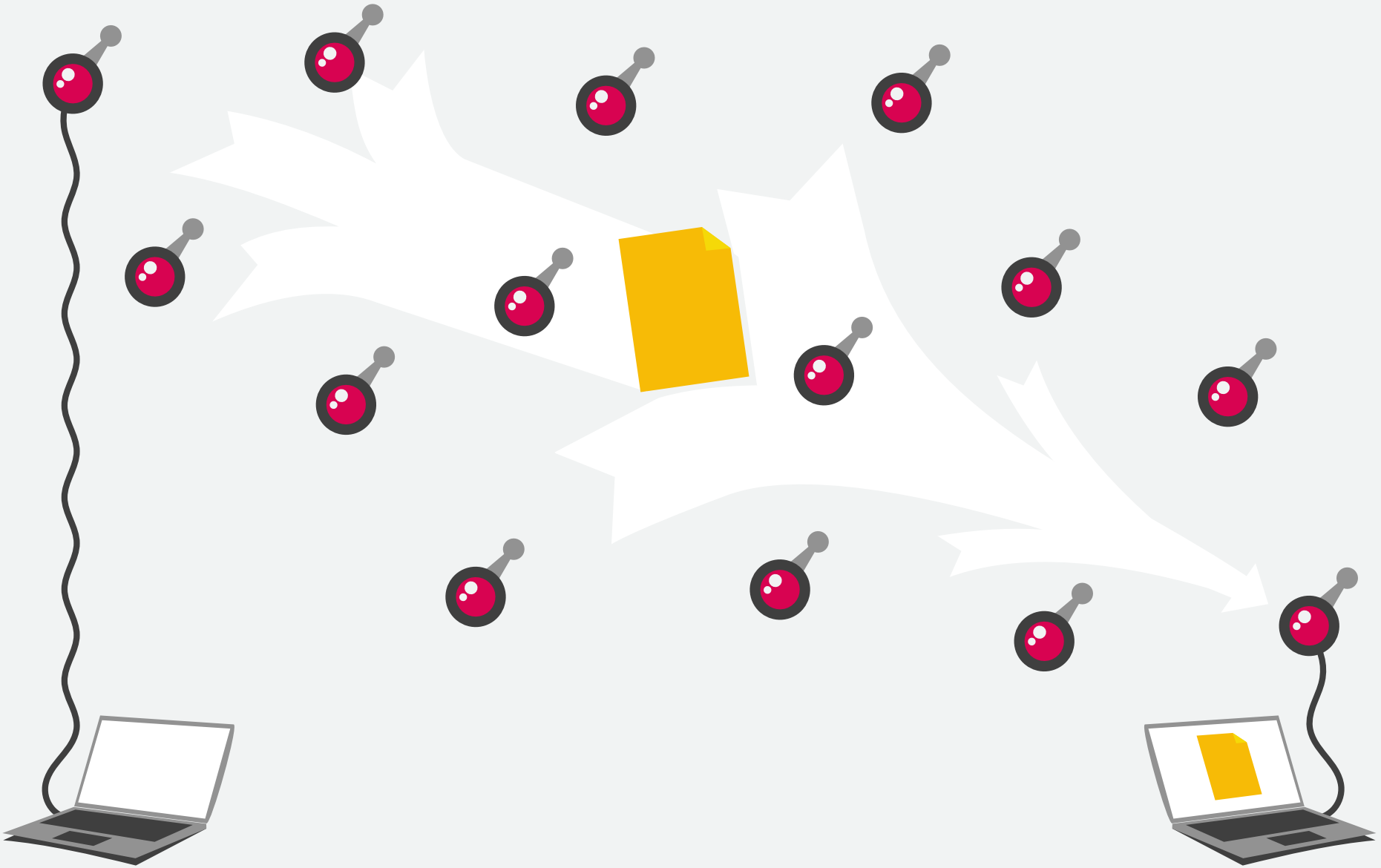

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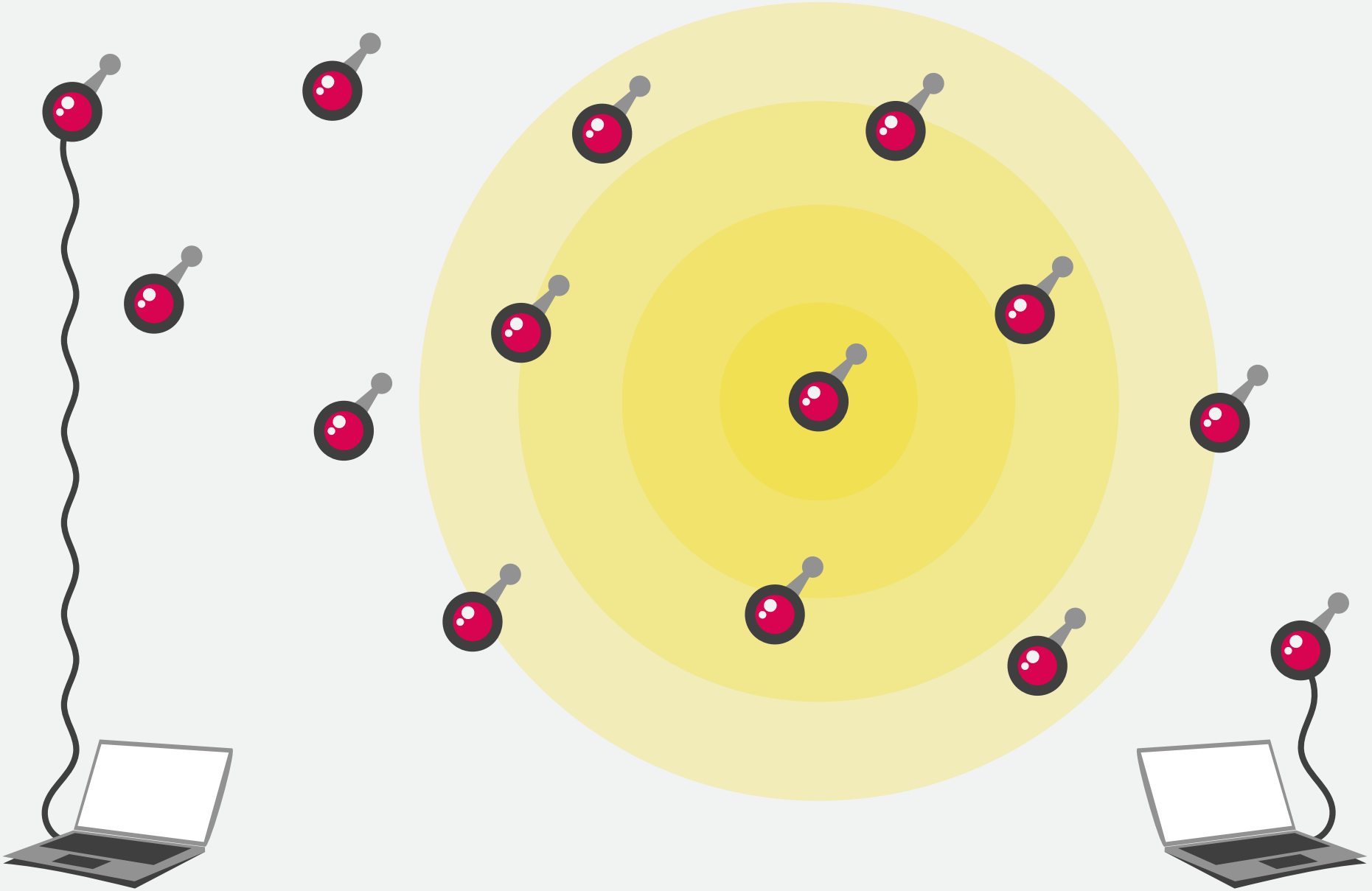

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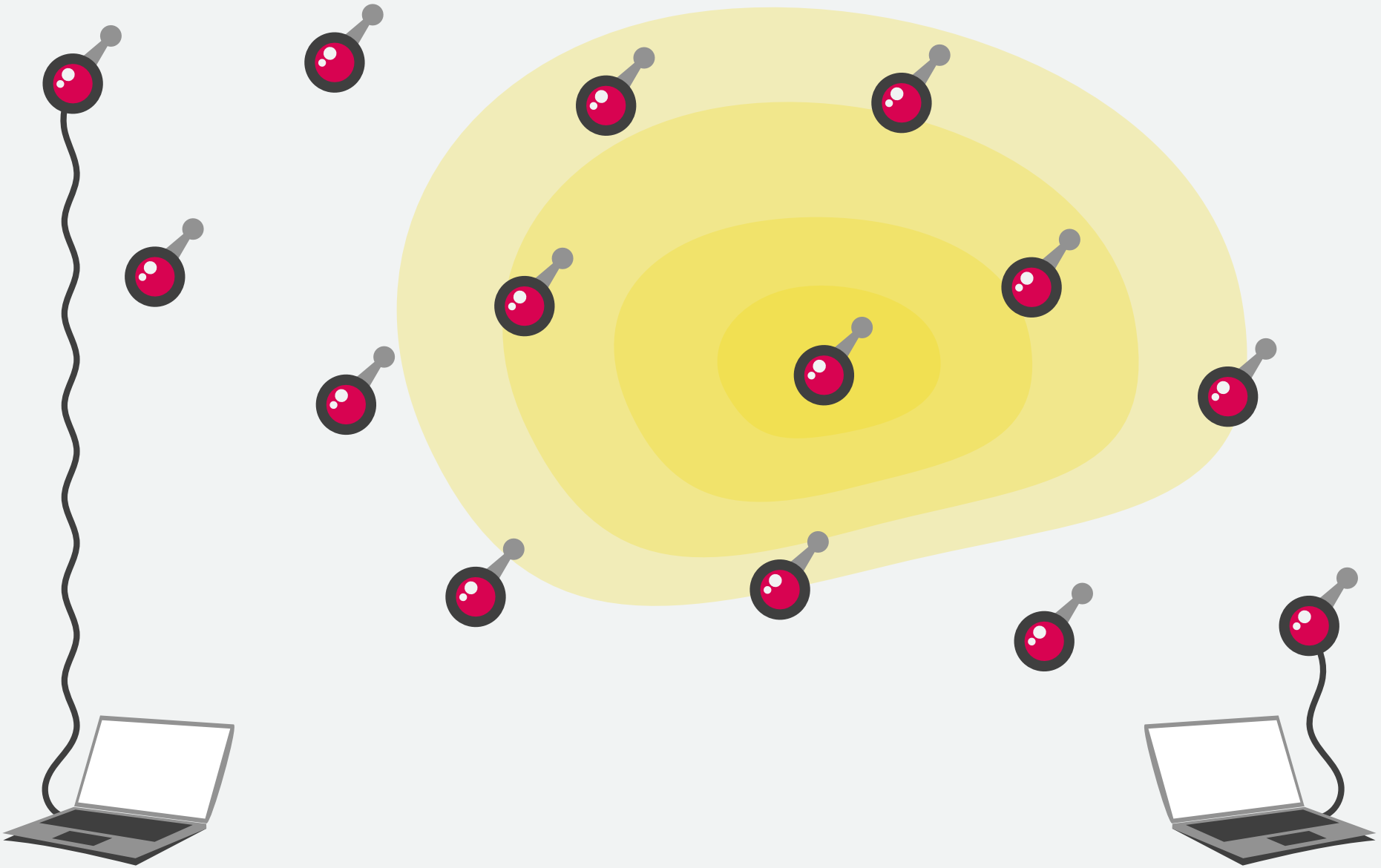

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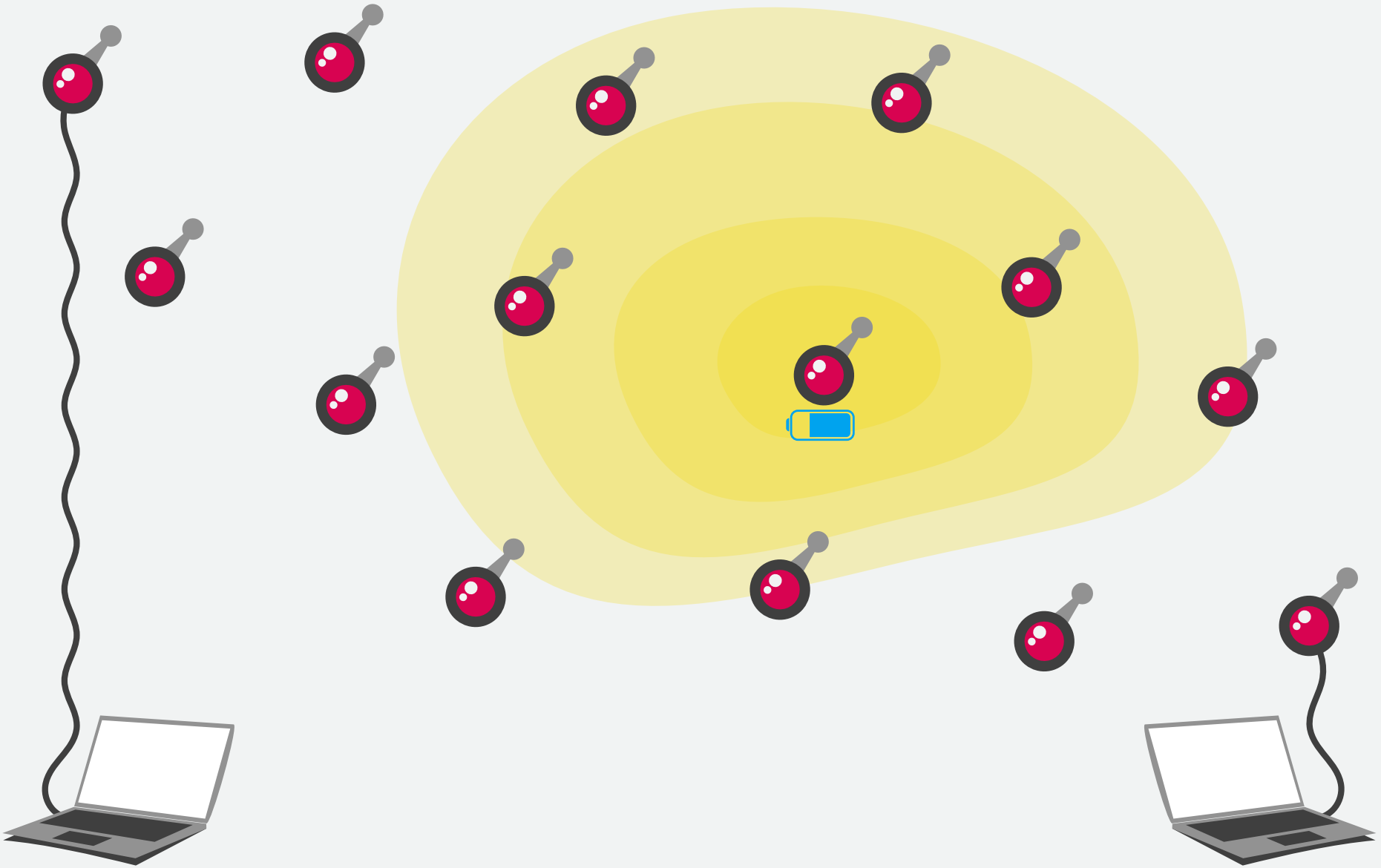

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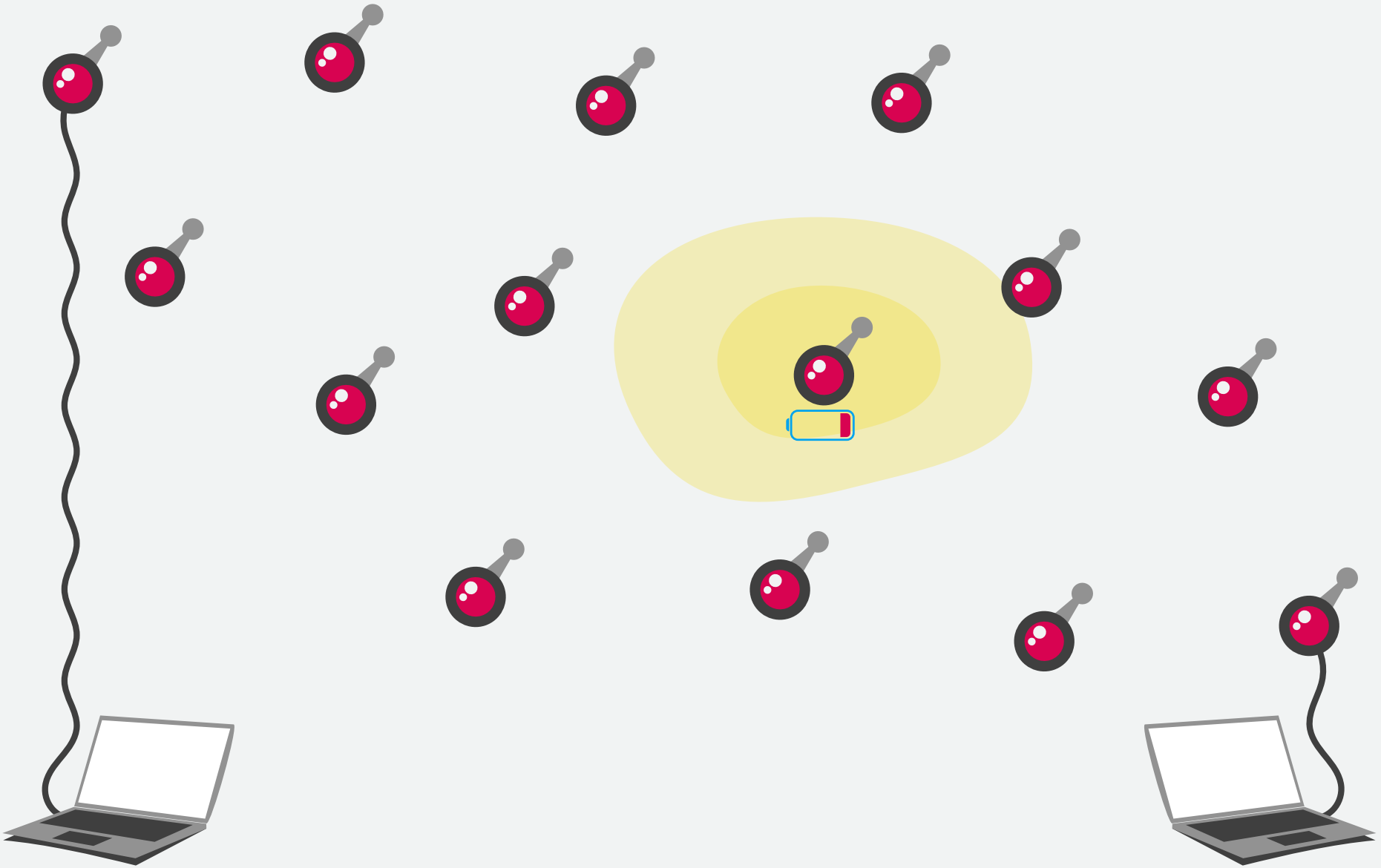

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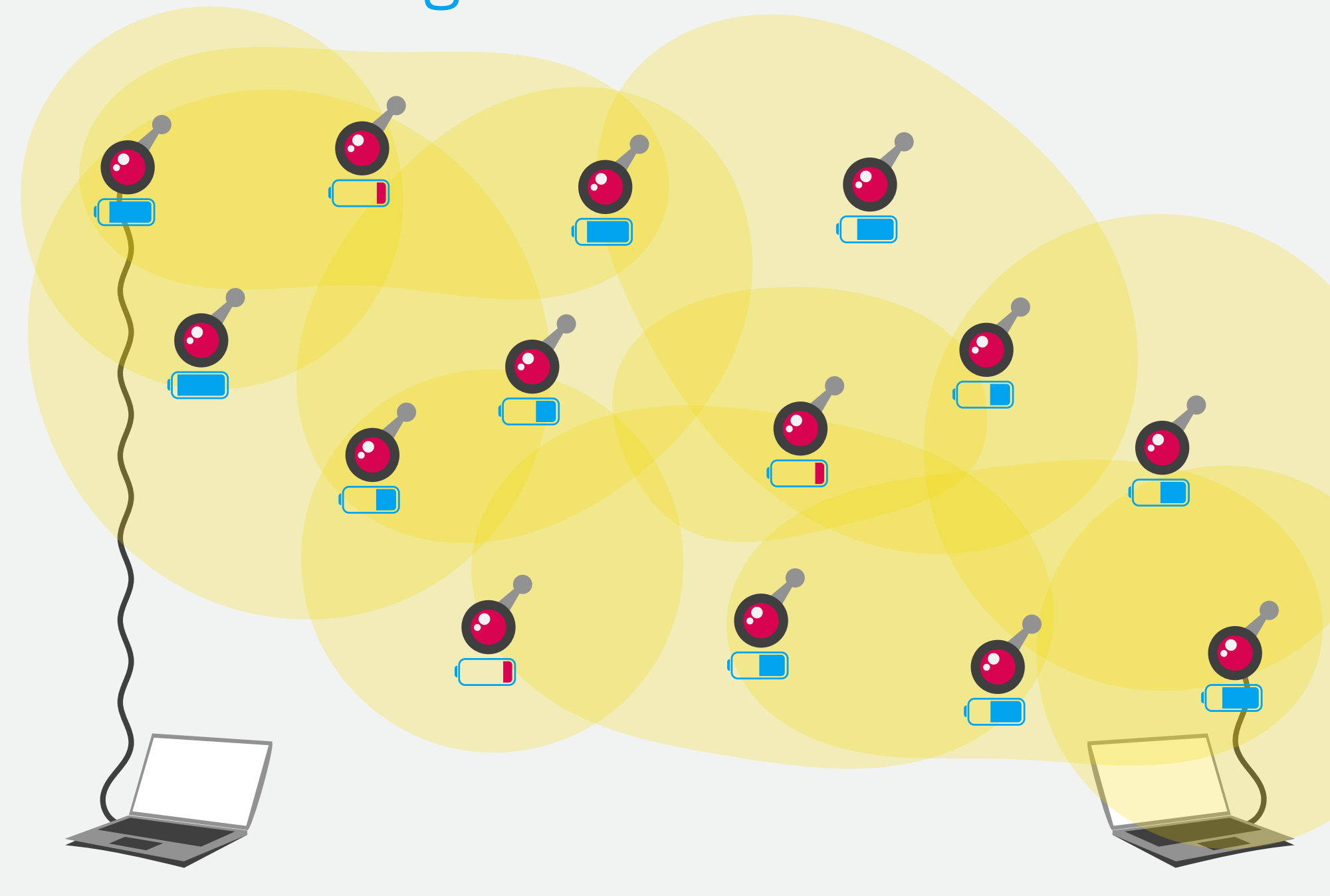

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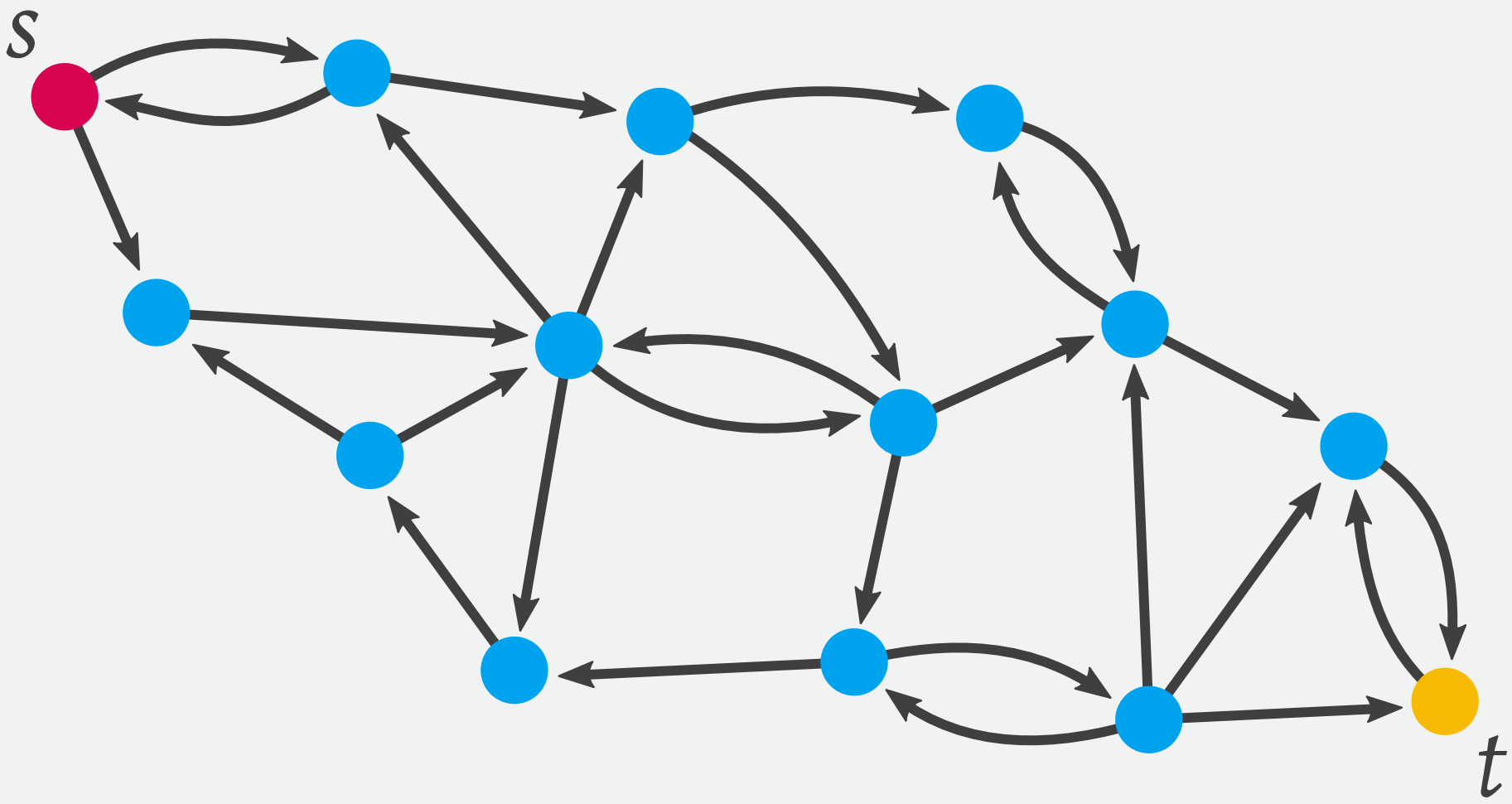

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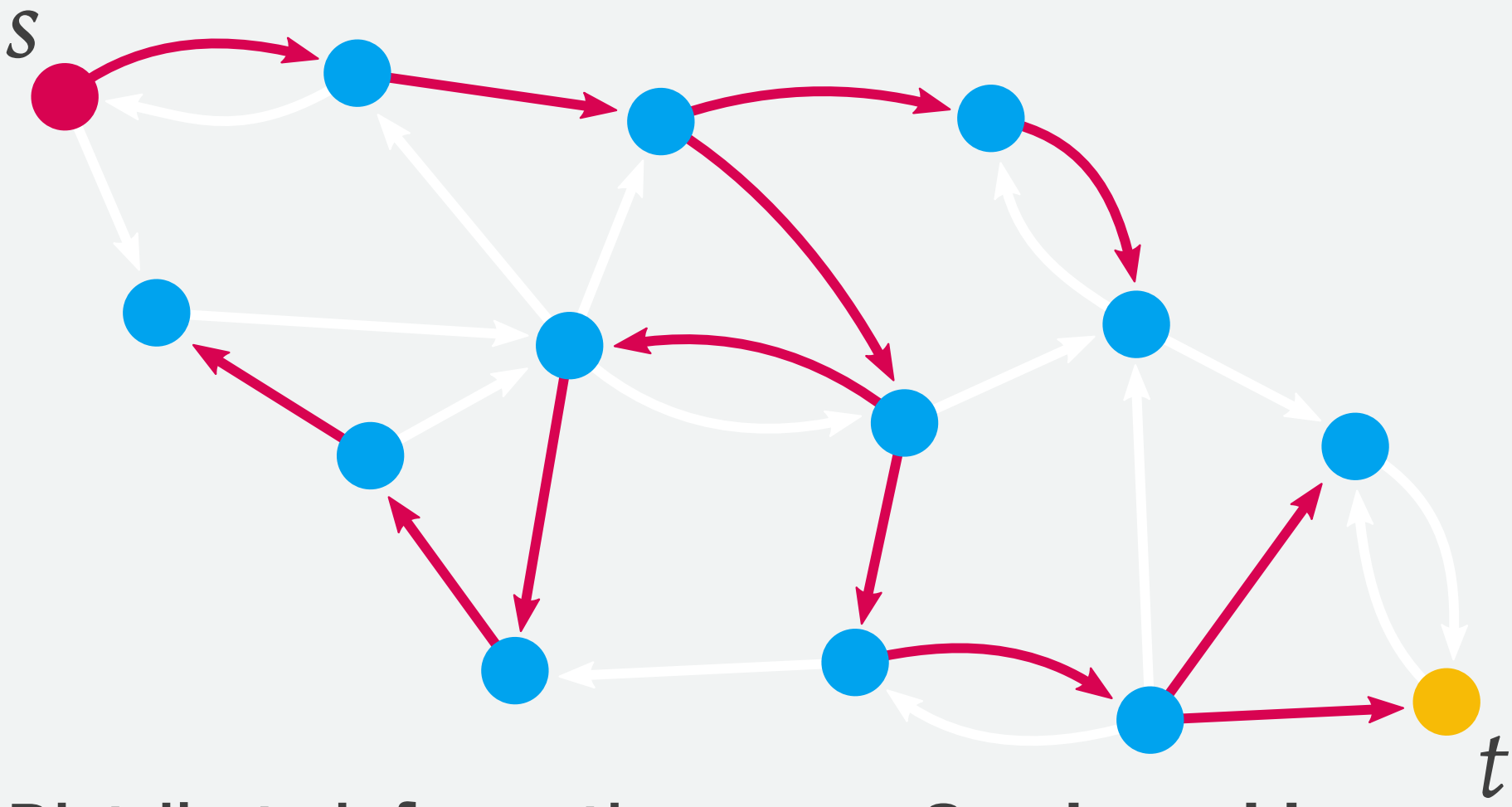
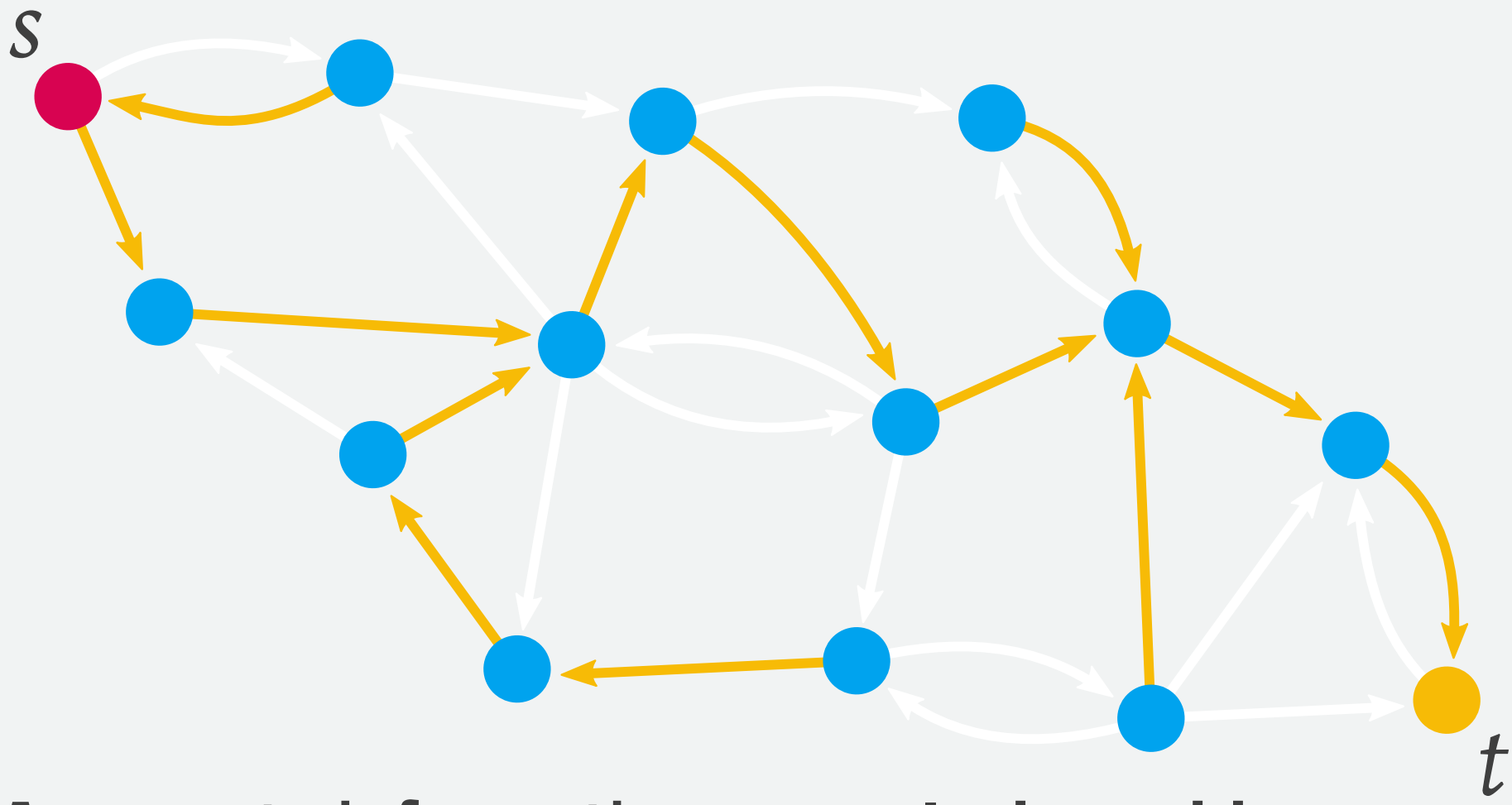


Distribute information from node s

≙

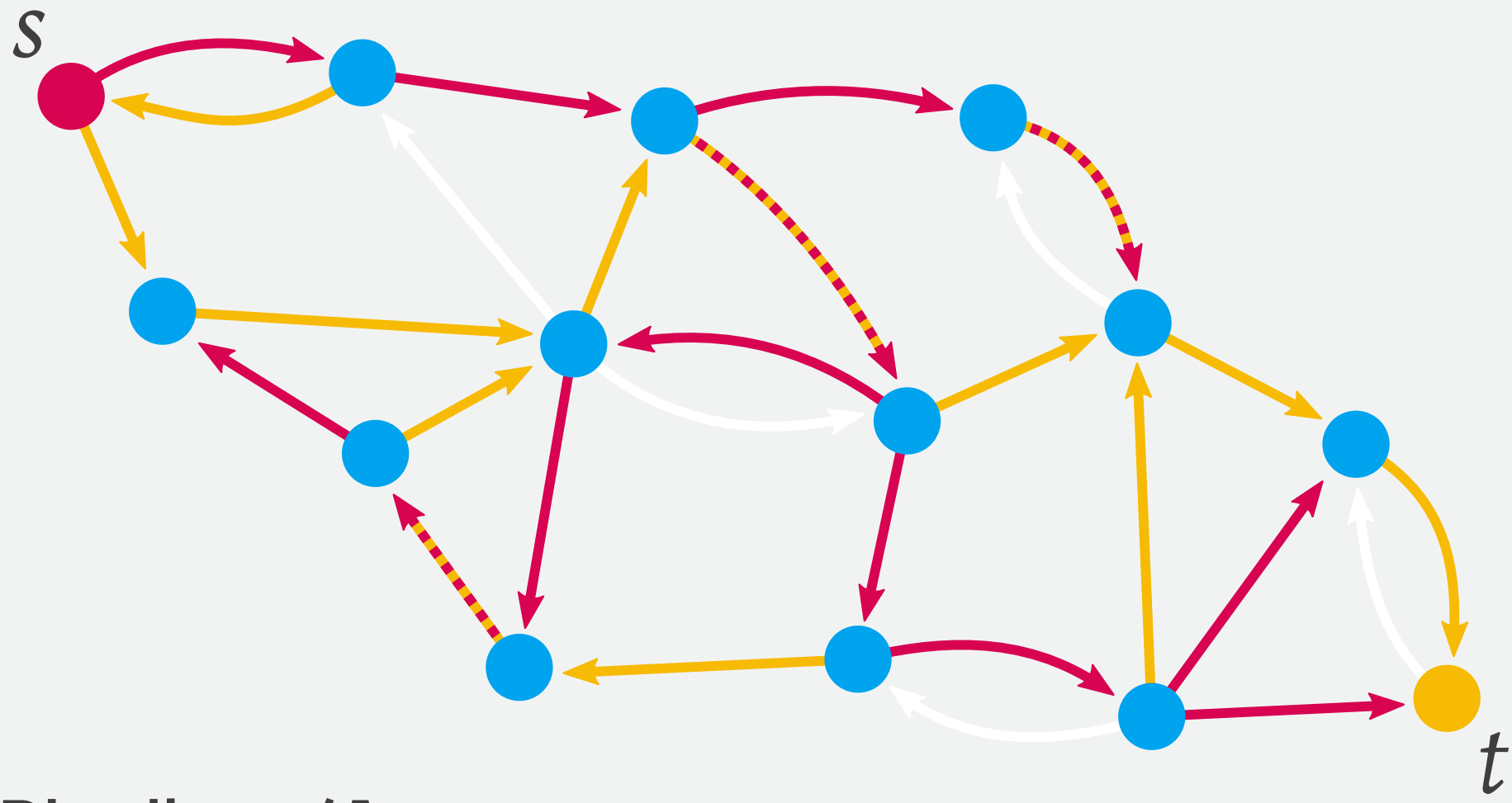
Out-branching with root s

Routing in Sensor Networks



Aggregate information at node t ≙ **In-branching with root t**

Routing in Sensor Networks



Distribute/Aggregate with little crosstalk

$\hat{=}$

In- and Out-branching with few common arcs

Distinct Branchings

An out-branching T^+ and and in-branching T^- are called k-distinct if $\left|A(T^+) \setminus A(T^-)\right| \geq k$.

k-DISTINCT BRANCHINGS

Input: A digraph D, an integer k.

Question: Does D contain k-distinct in- and out-branchings?

Previous work

- **NP-complete to decide whether arc-disjoint in- and out-branchings exist**

  Bang-Jensen J. Edge-disjoint in-and out-branchings in tournaments and related path problems.
  Journal of Combinatorial Theory, Series B. 1991 Jan 1;51(1):1-23.

- **FPT in strong digraphs, parameterized by k**

  Bang-Jensen J, Saurabh S, Simonsen S. Parameterized algorithms for non-separating trees and branchings in digraphs.
  Algorithmica. 2016 Sep 1;76(1):279-96.

- **FPT in general digraphs?**

- **FPT if $s = t$?** (SINGLE ROOT k-DISTINCT BRANCHINGS)

  Bang-Jensen J, Yeo A. The minimum spanning strong subdigraph problem is fixed parameter tractable.
  Discrete Applied Mathematics. 2008 Aug 6;156(15):2924-9.

Our result

ROOTED k-DISTINCT BRANCHINGS

Input: A digraph D, integer k, vertices s, t.

Question: Does D contain k-distinct in- and out-branchings rooted at s and t respectively?

Theorem.
ROOTED k-DISTINCT BRANCHINGS **is in** FPT.

Schematic of our result

- **Preprocessing:**
  - **Every arc appears in some rooted in- or out-branching**
  - **D is strongly connected, thus we can search for rooted in- and out-trees**

incurs factor 2 in parameter

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- **If D has no out-tree with many leaves, it has bounded pathwidth**

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Remaining case:

Out-tree with many leaves, but incorrect root

Good case: Many leaves

Lemma.

Assume D contains an in- and out-branching. If D contains an out-branching T^+ with at least $k+1$ leaves, then every in-branching T^- of D is k-distinct from T^+.

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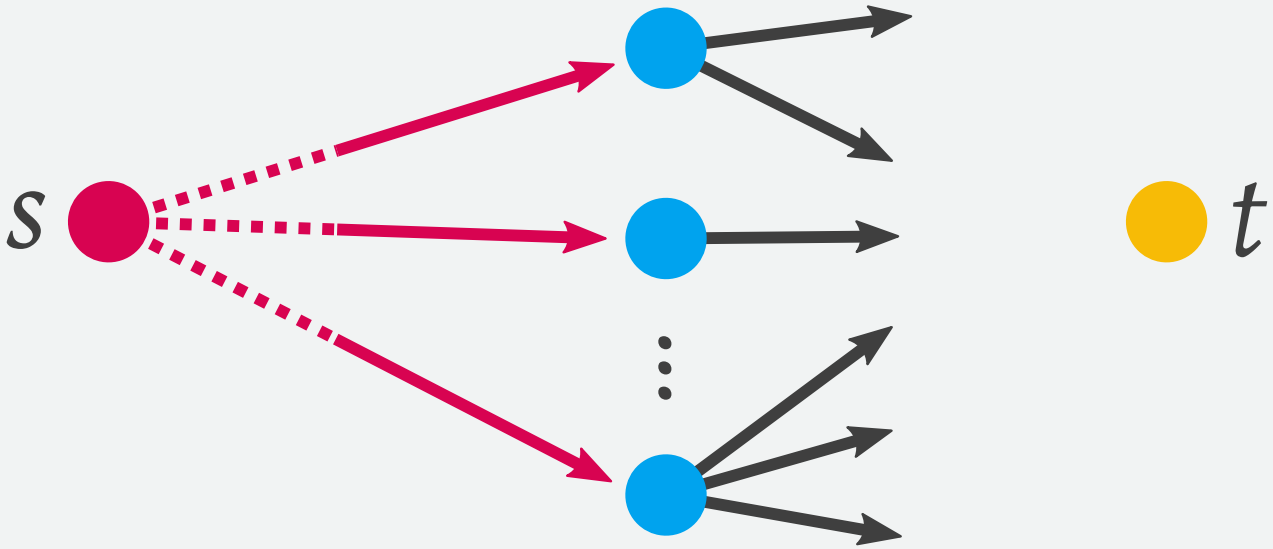

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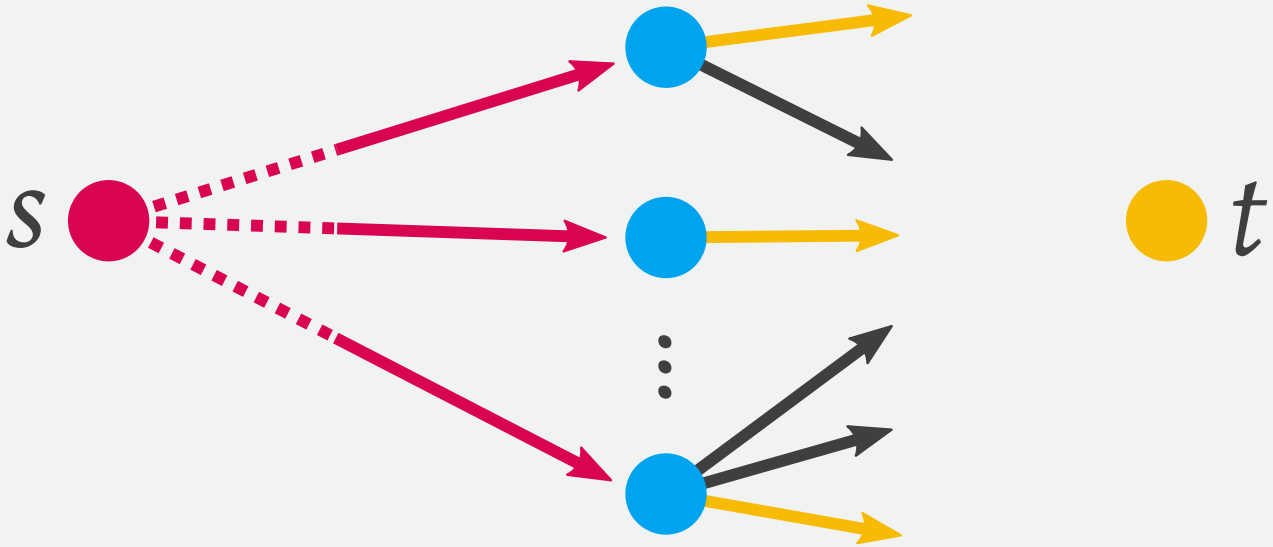

Good case: high connectivity

Toy Lemma.
Let T^+ be an out-tree rooted at v with l leaves and assume that every vertex is bi-reachable from s. Then there exists on out-tree rooted at s with at least $l/2$ leaves.

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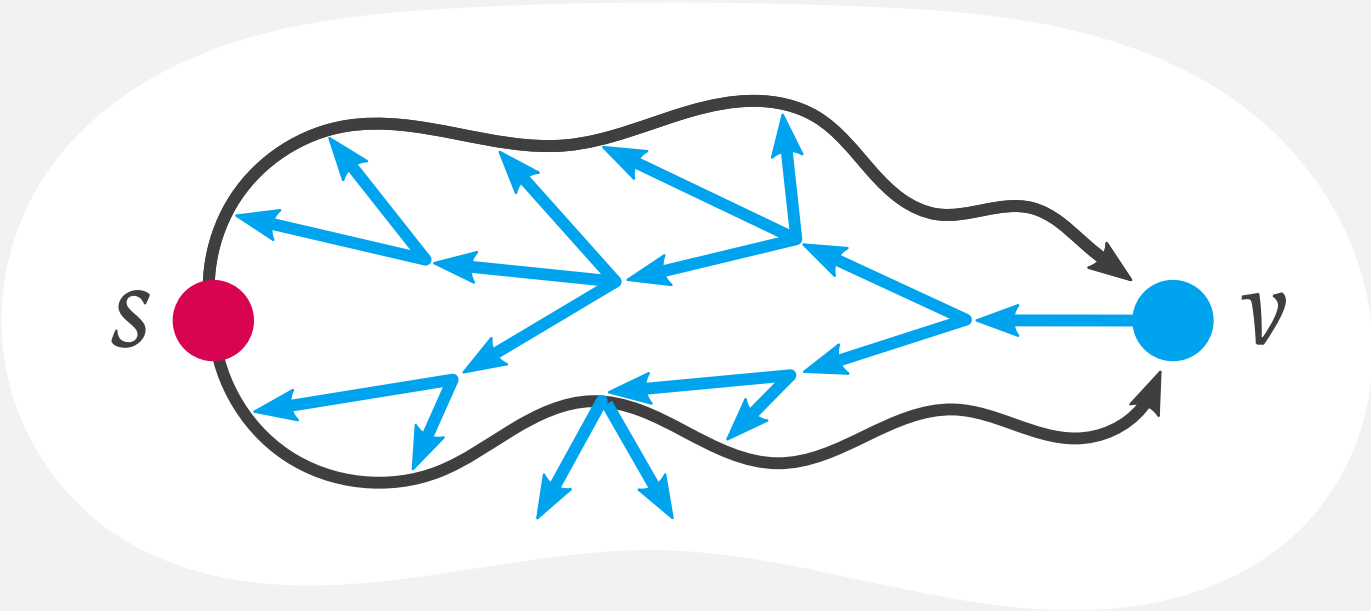

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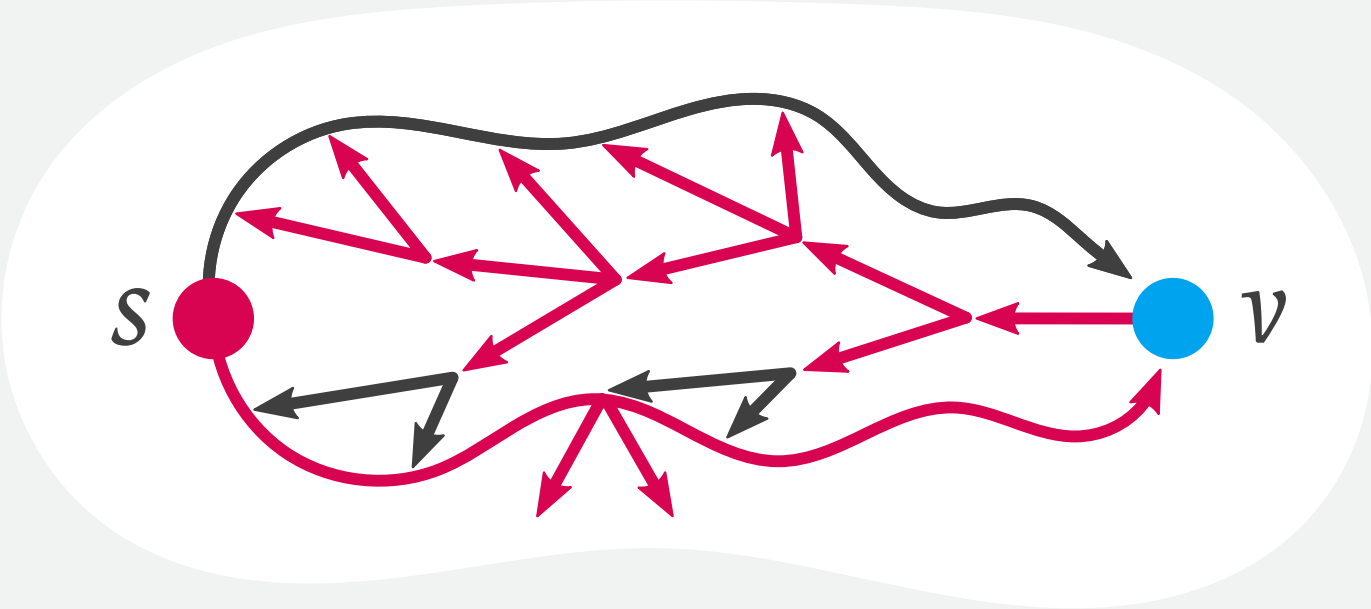
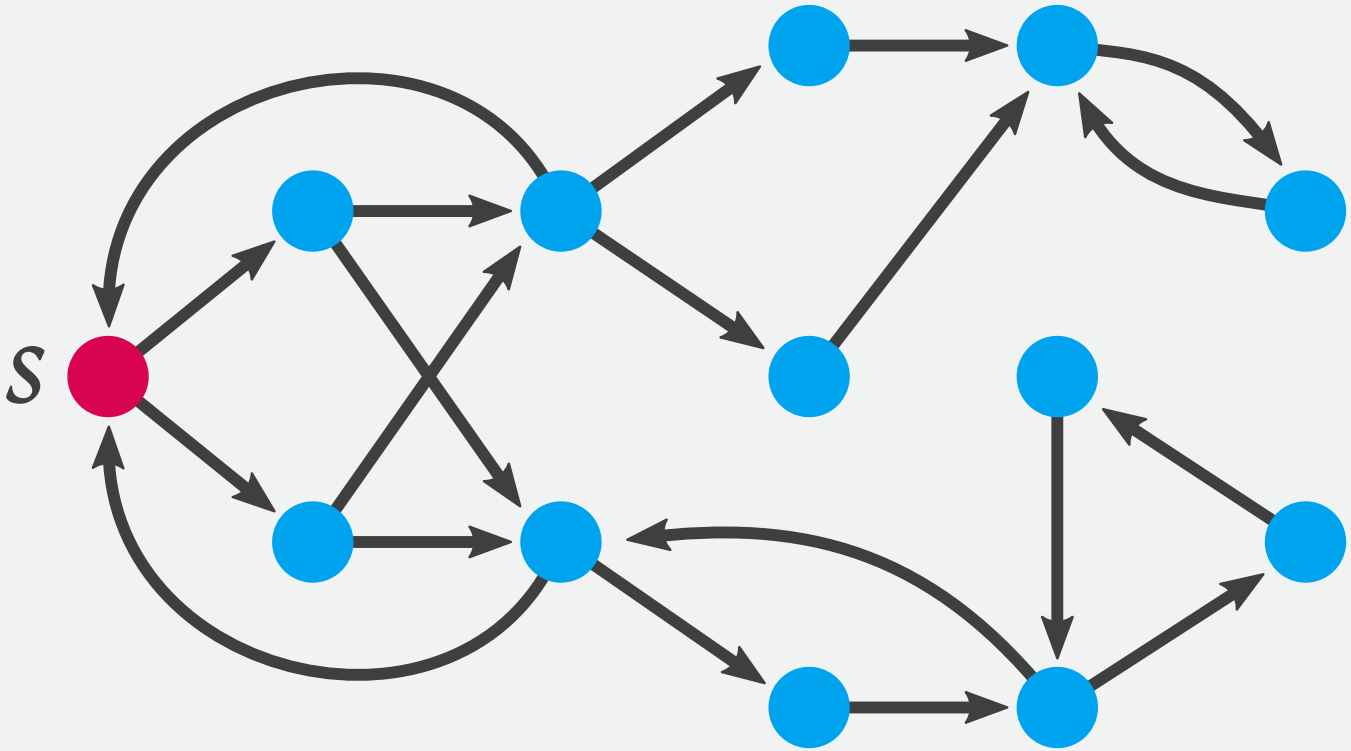


Decomposing along biconnectvity

Definition (Diblock).
The **diblock** of a vertex r is the union of $N^+(r)$ and all vertices that are bi-reachable from r.



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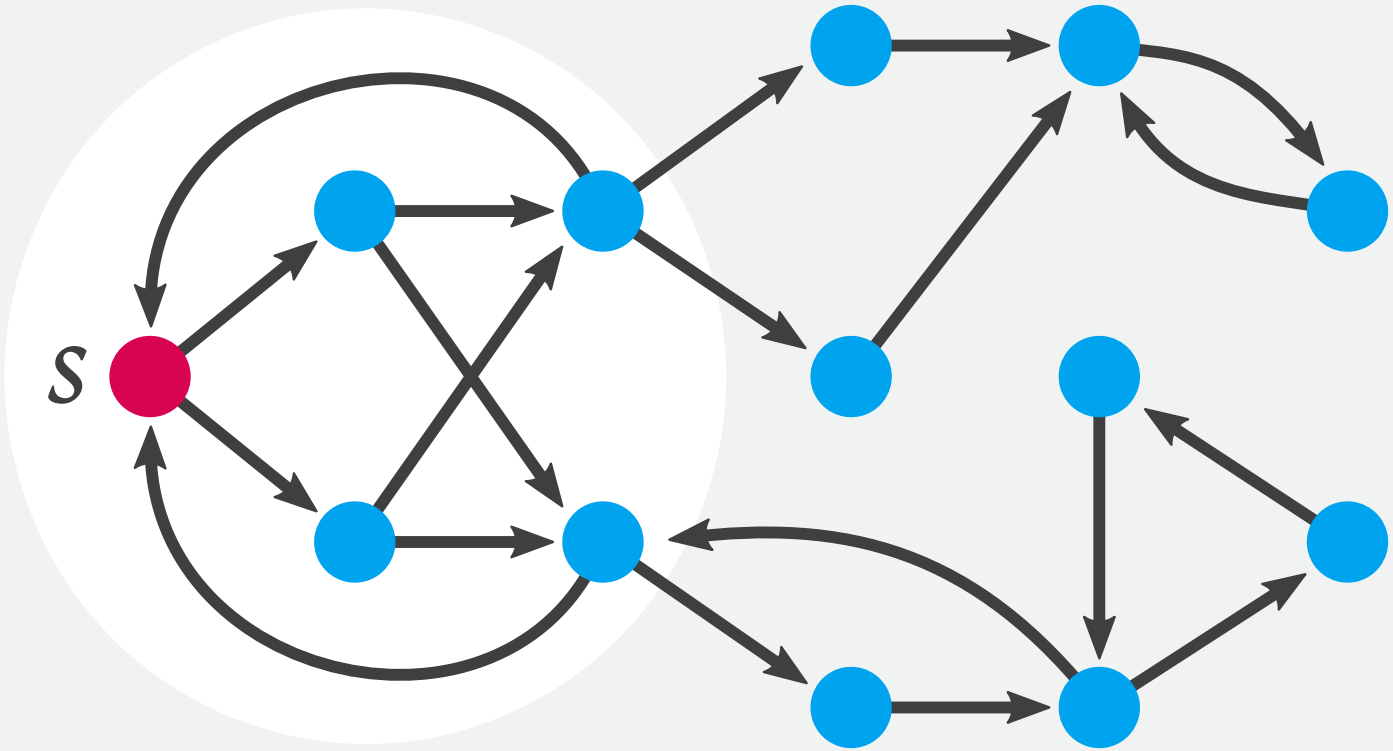

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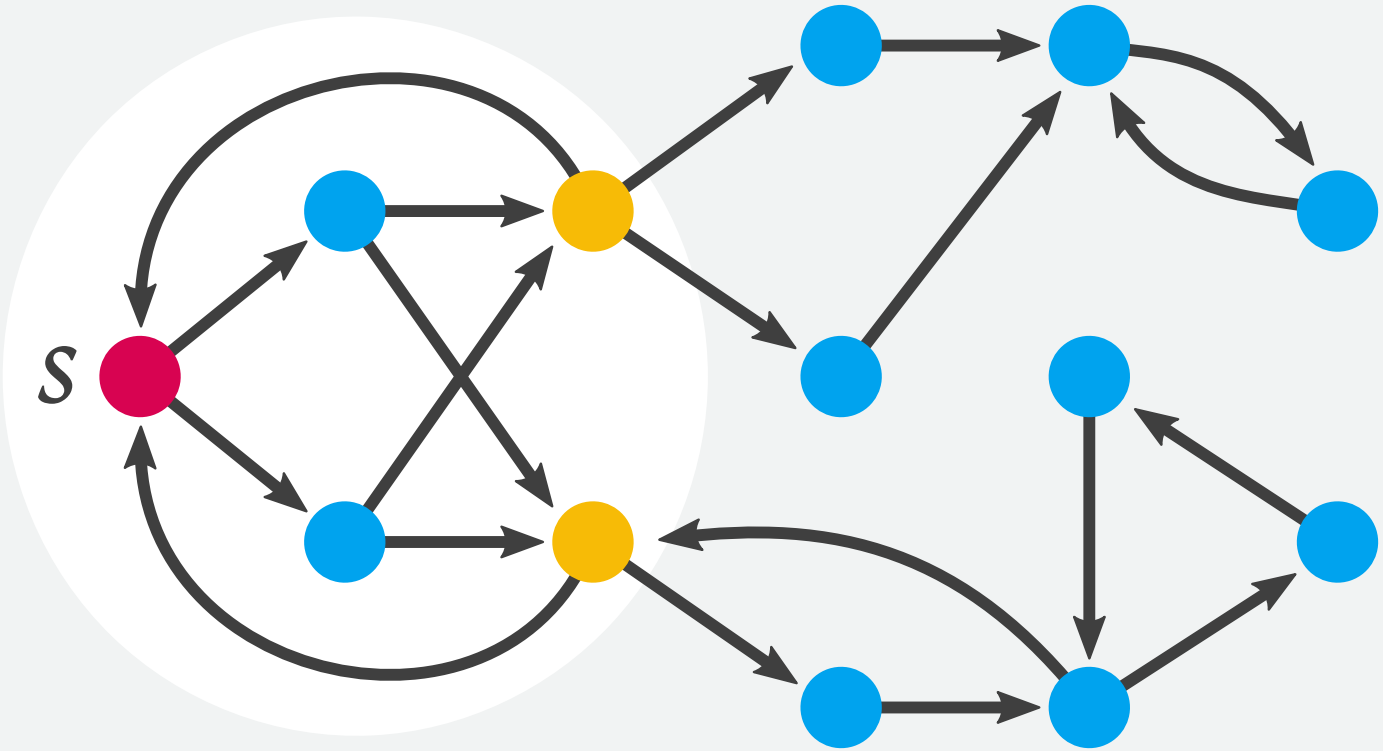
Definition (Cut decomposition).
Recursively decompose into diblocks and **bottlenecks**.



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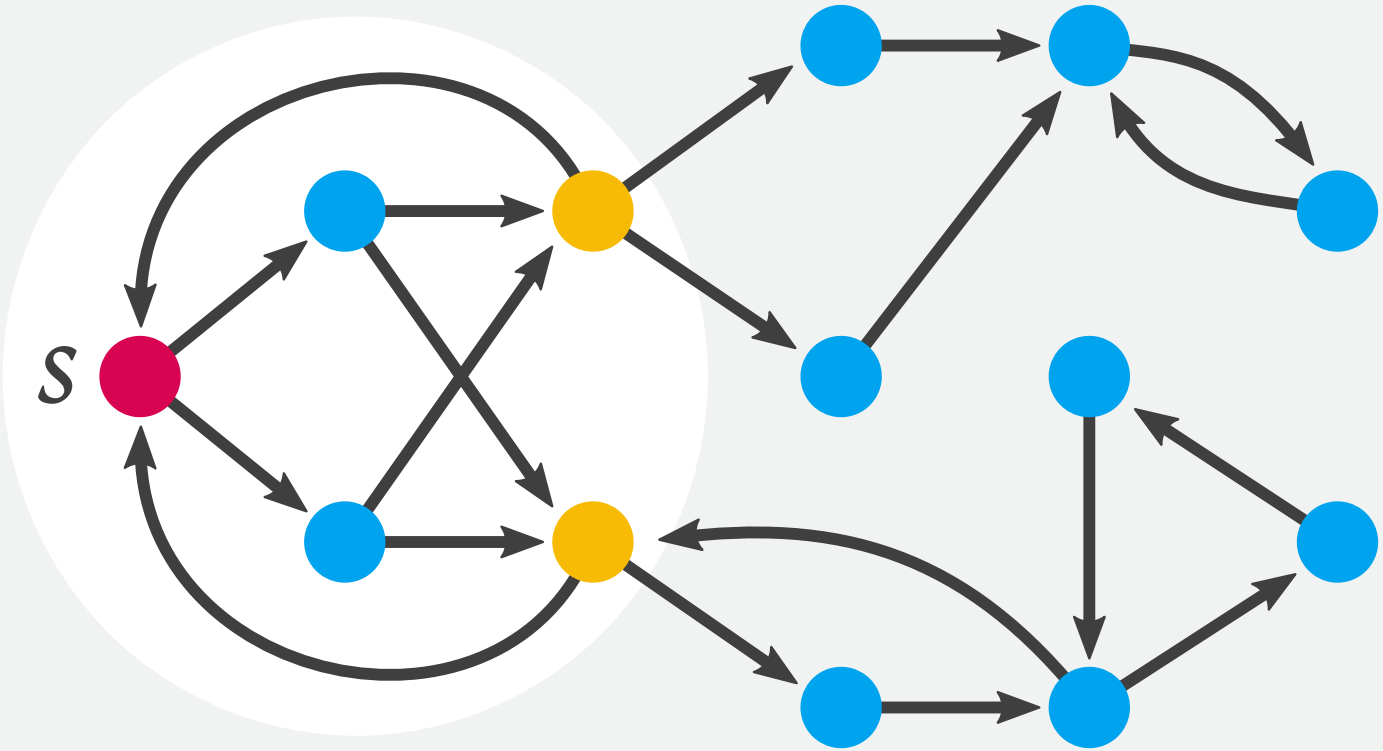

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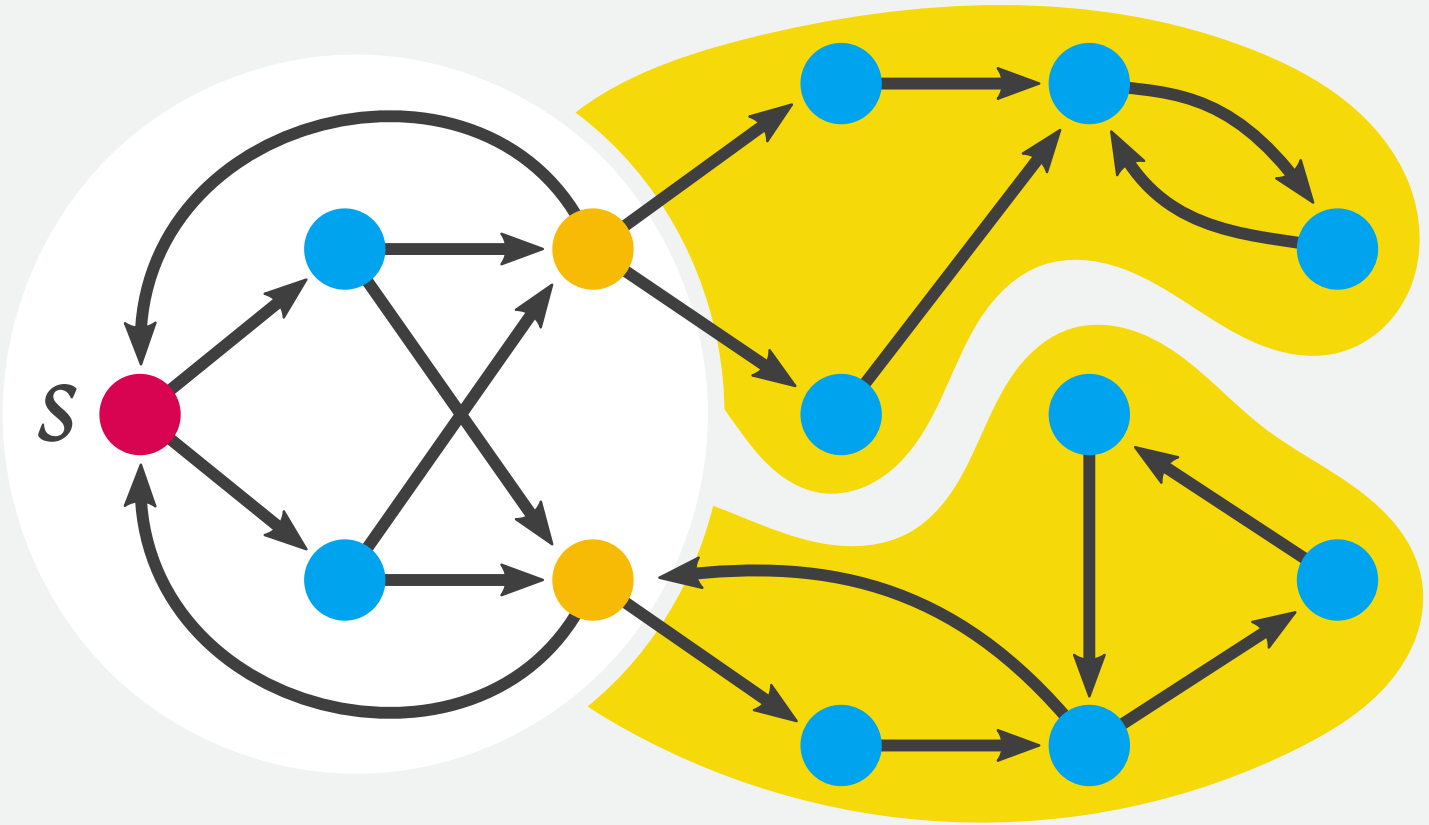

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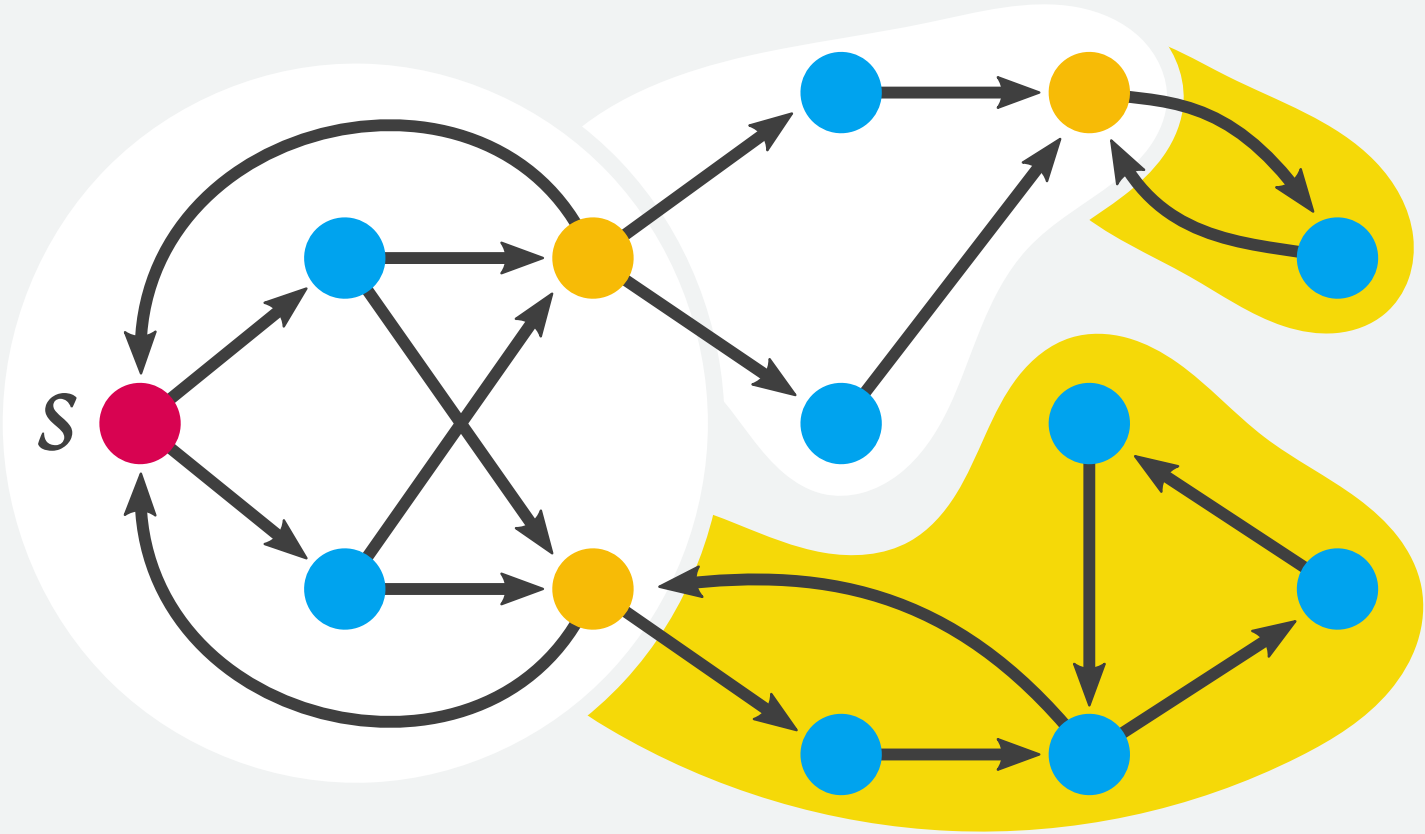

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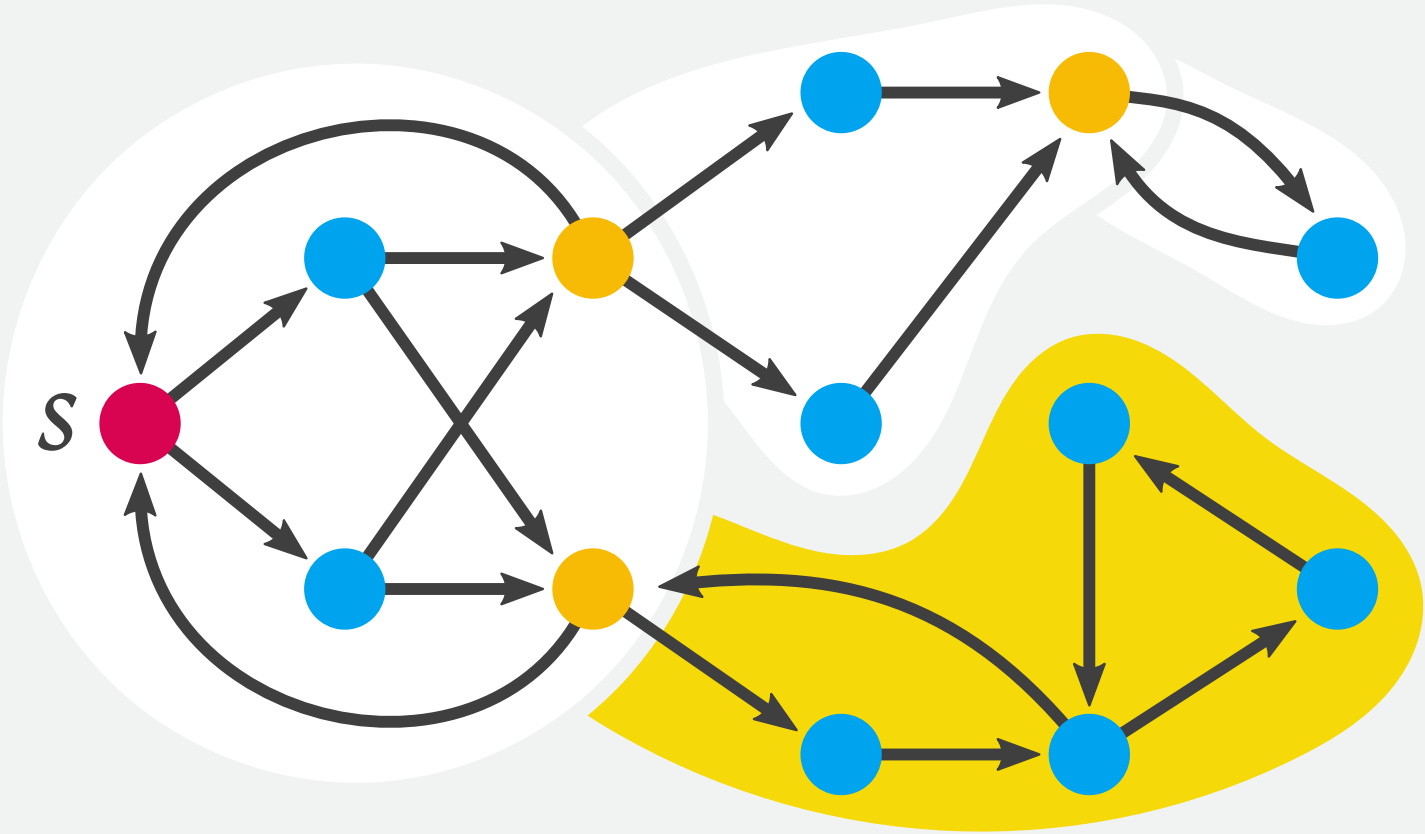

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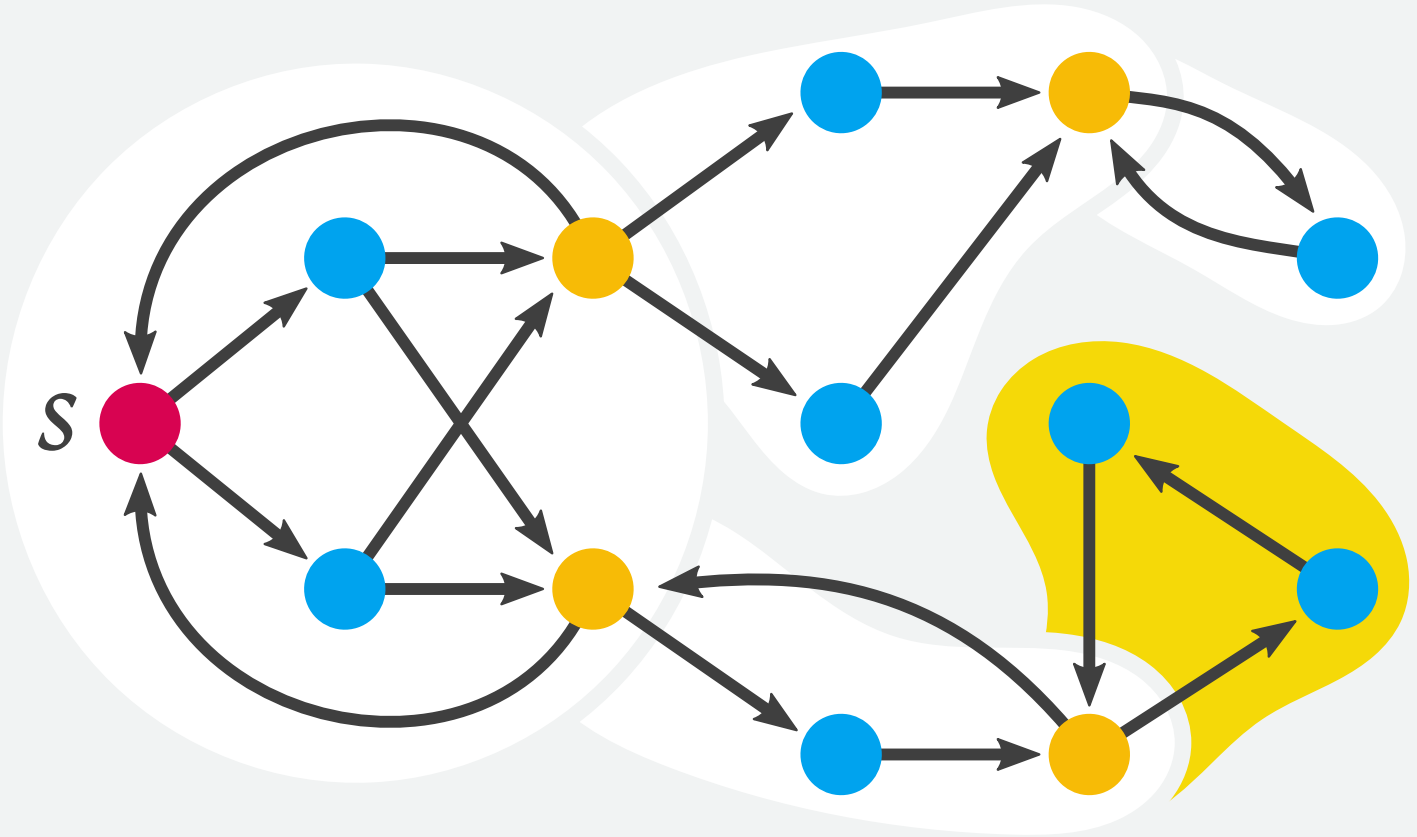

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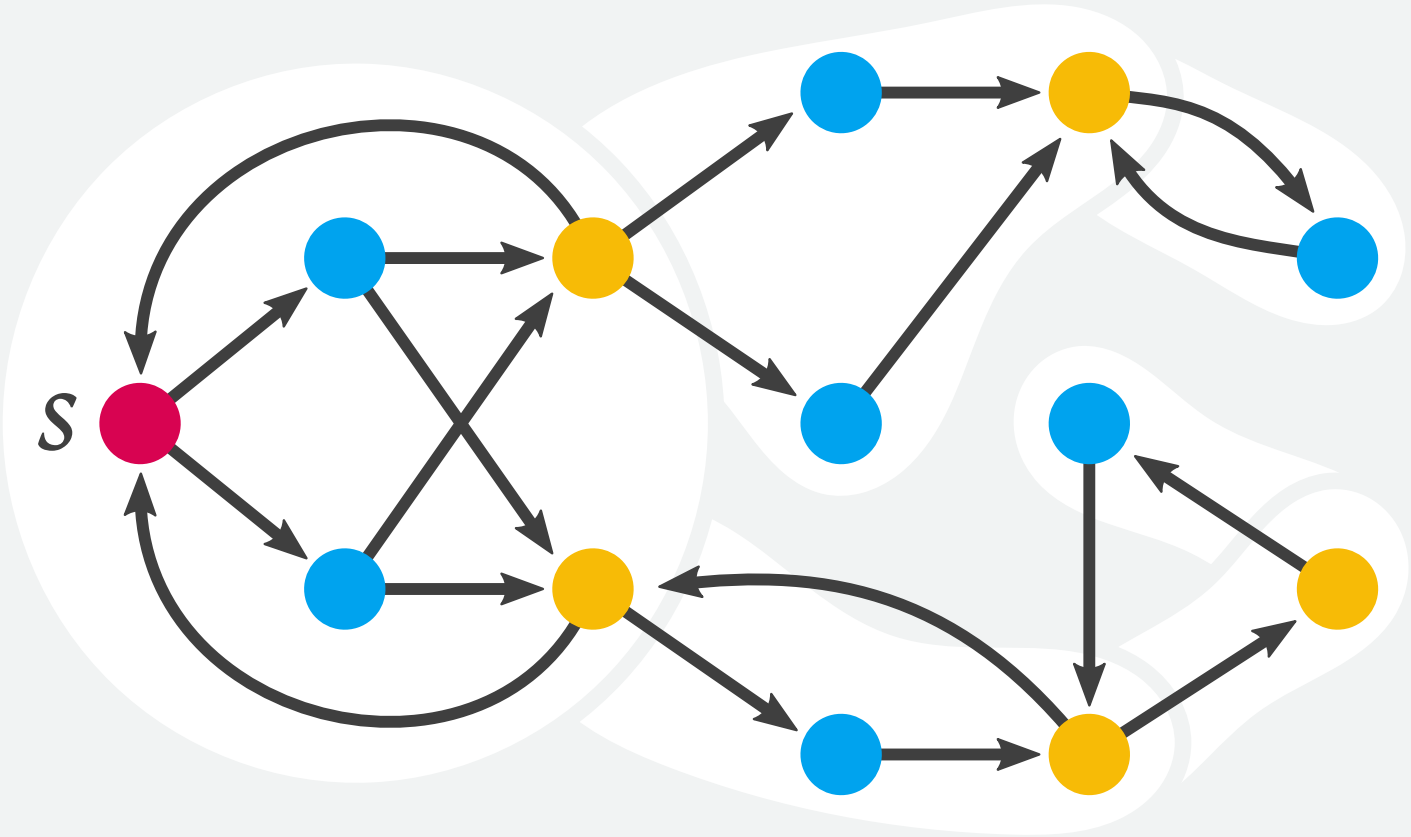

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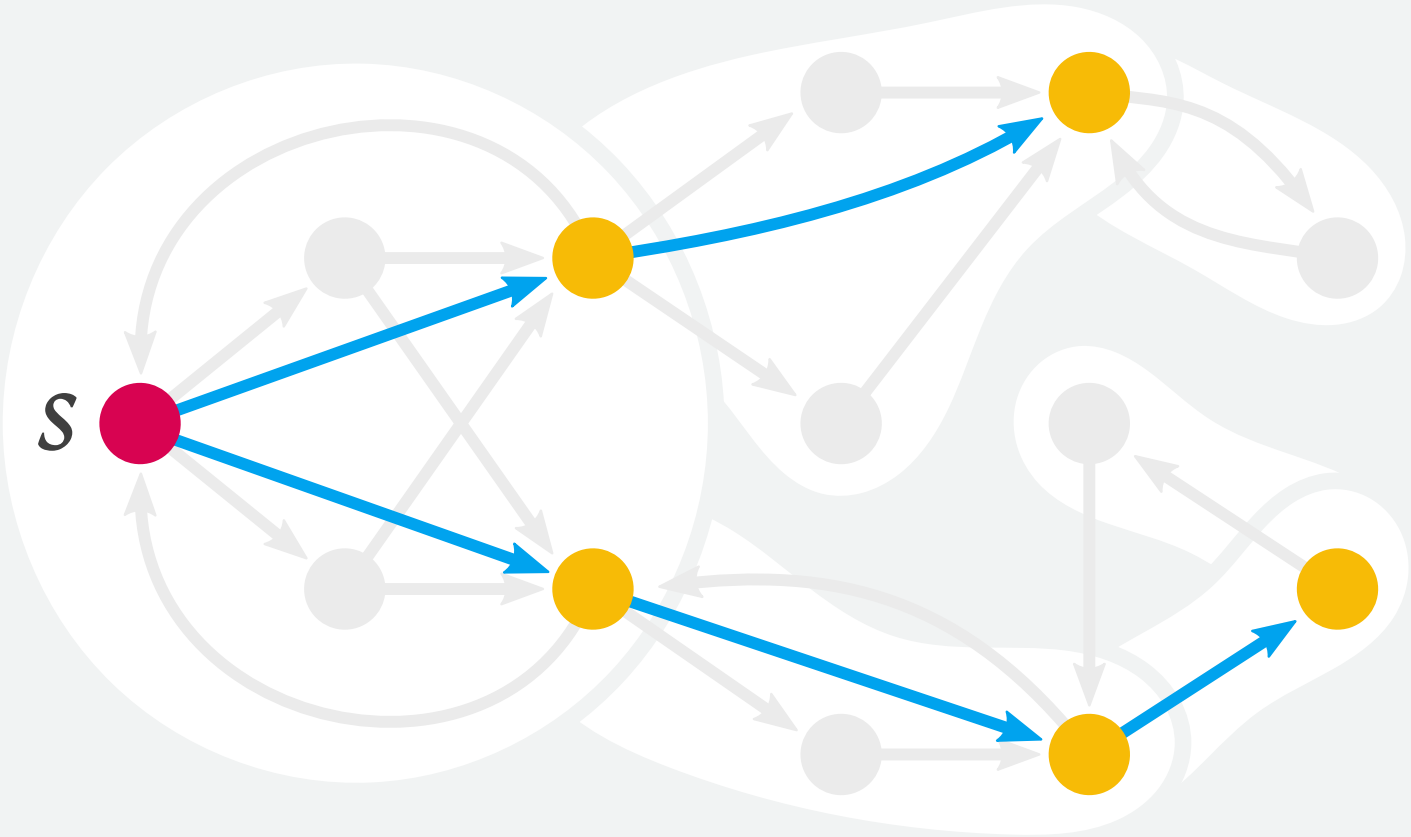

Re-rooting an out-tree

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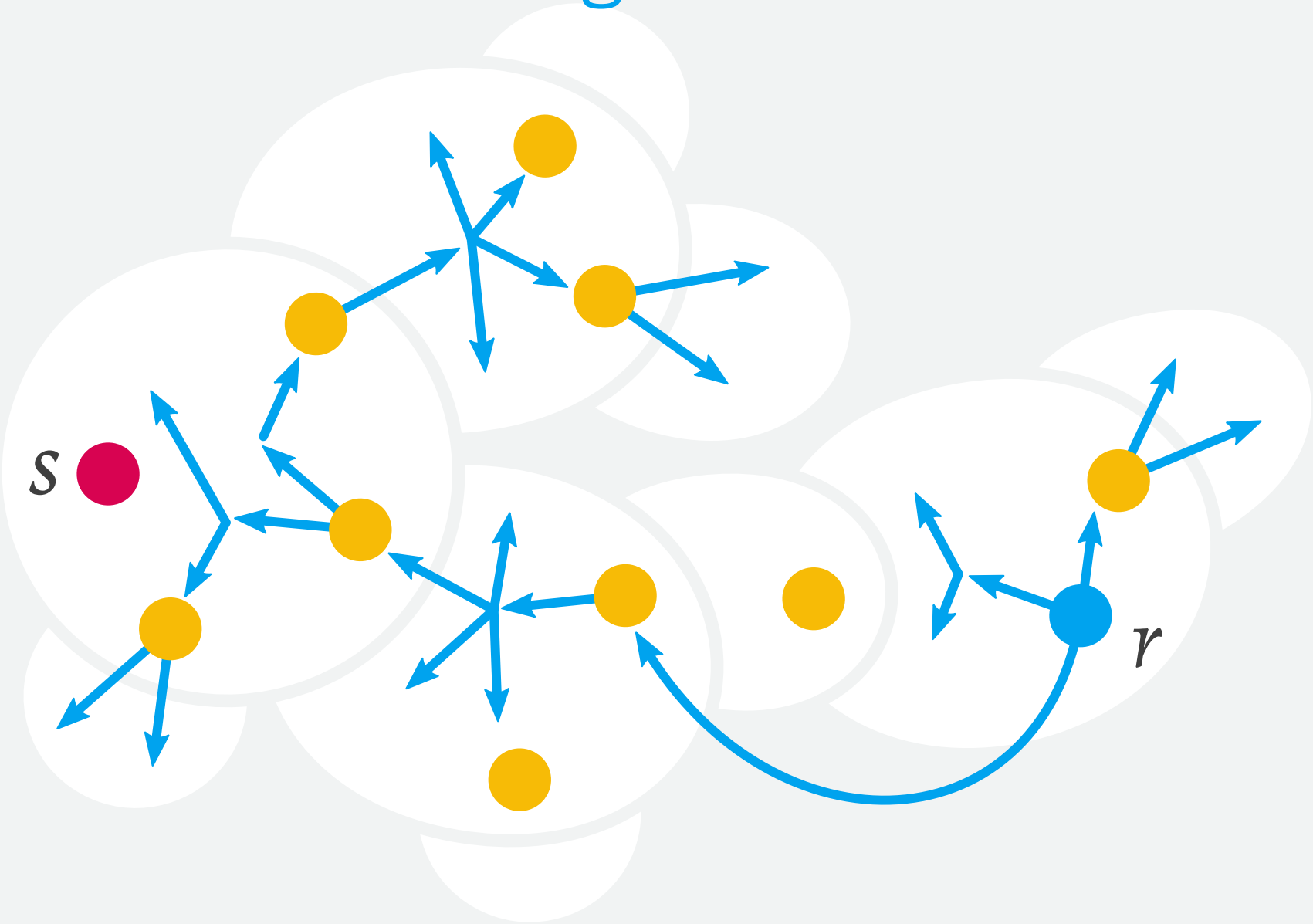

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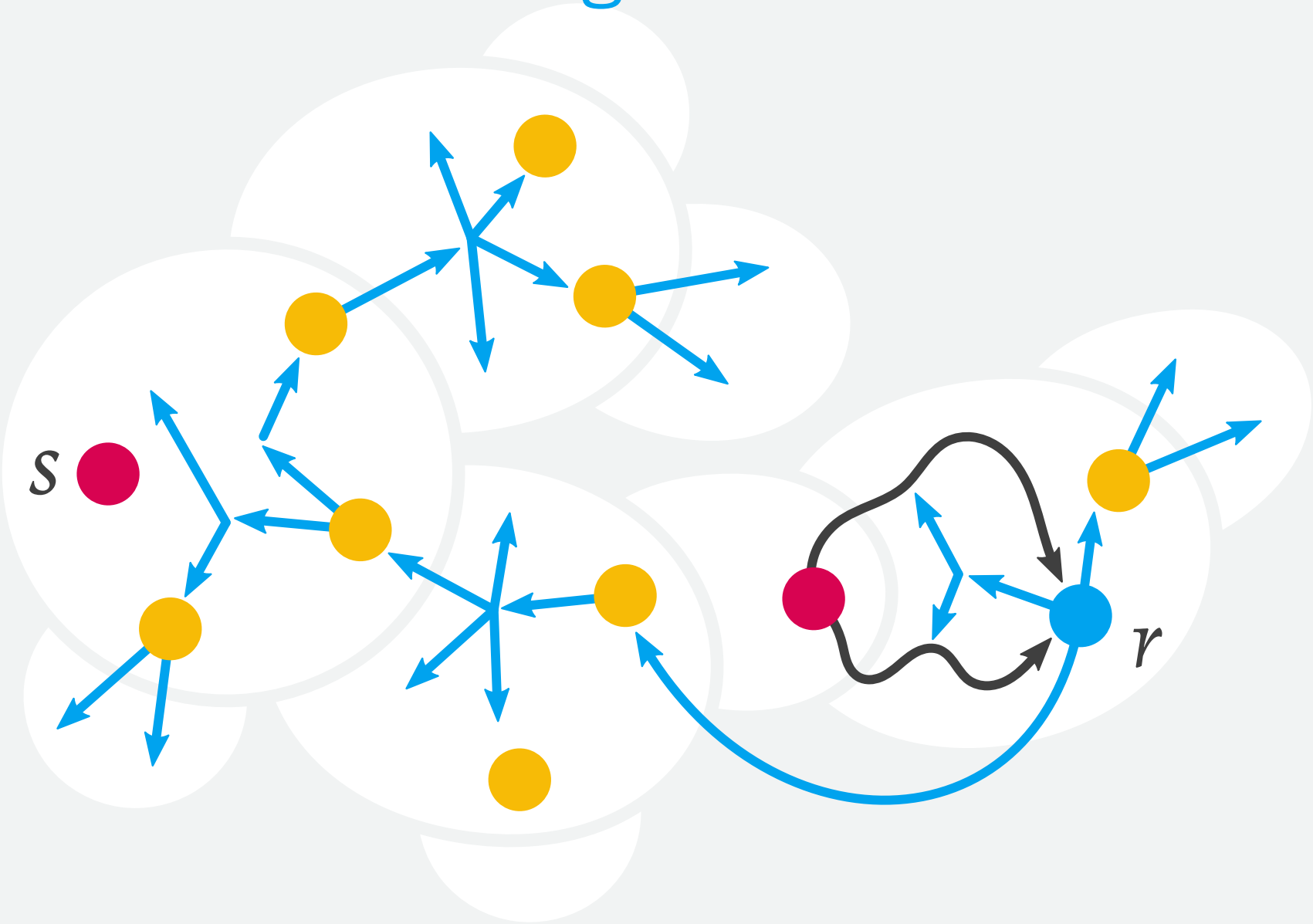

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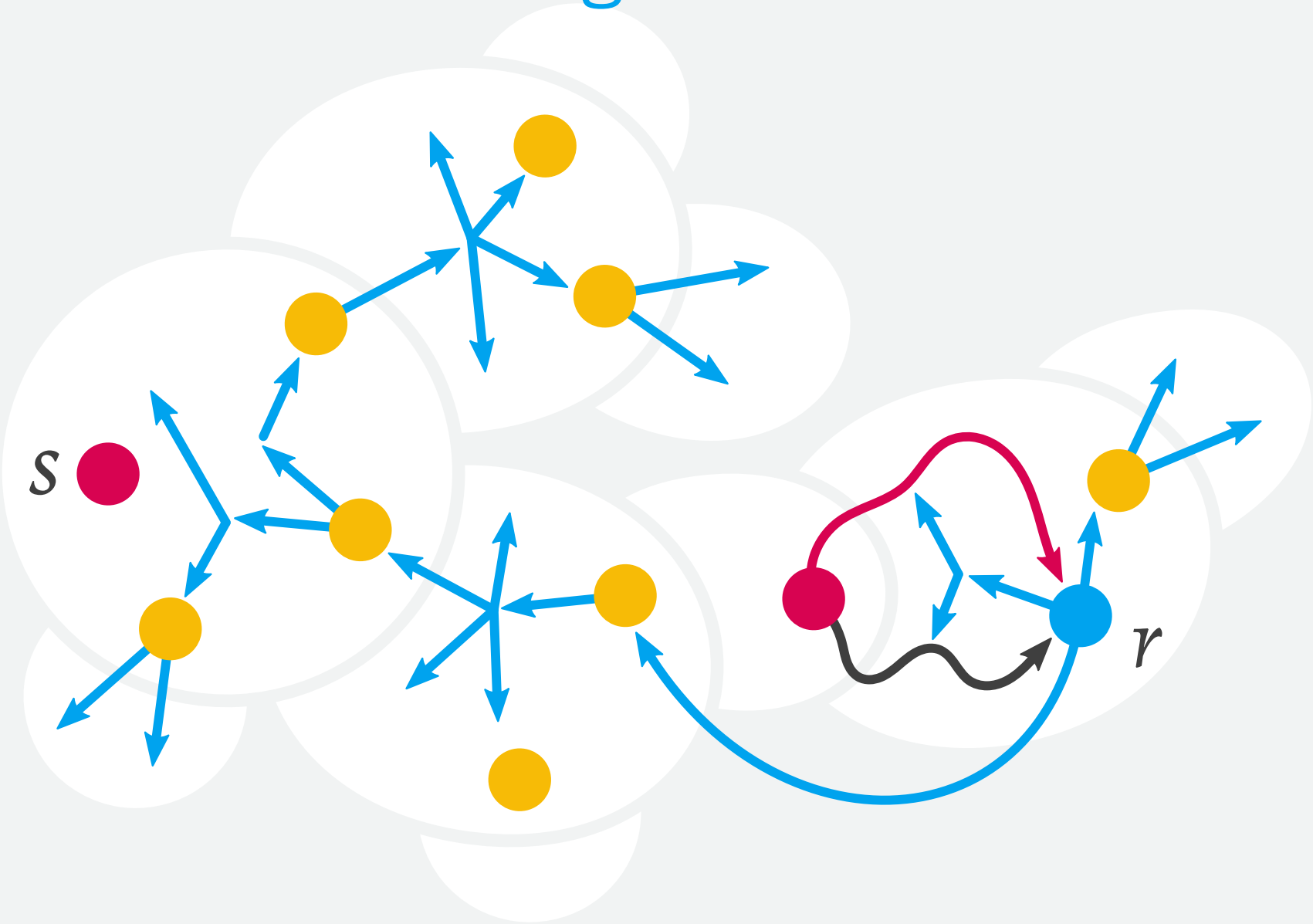

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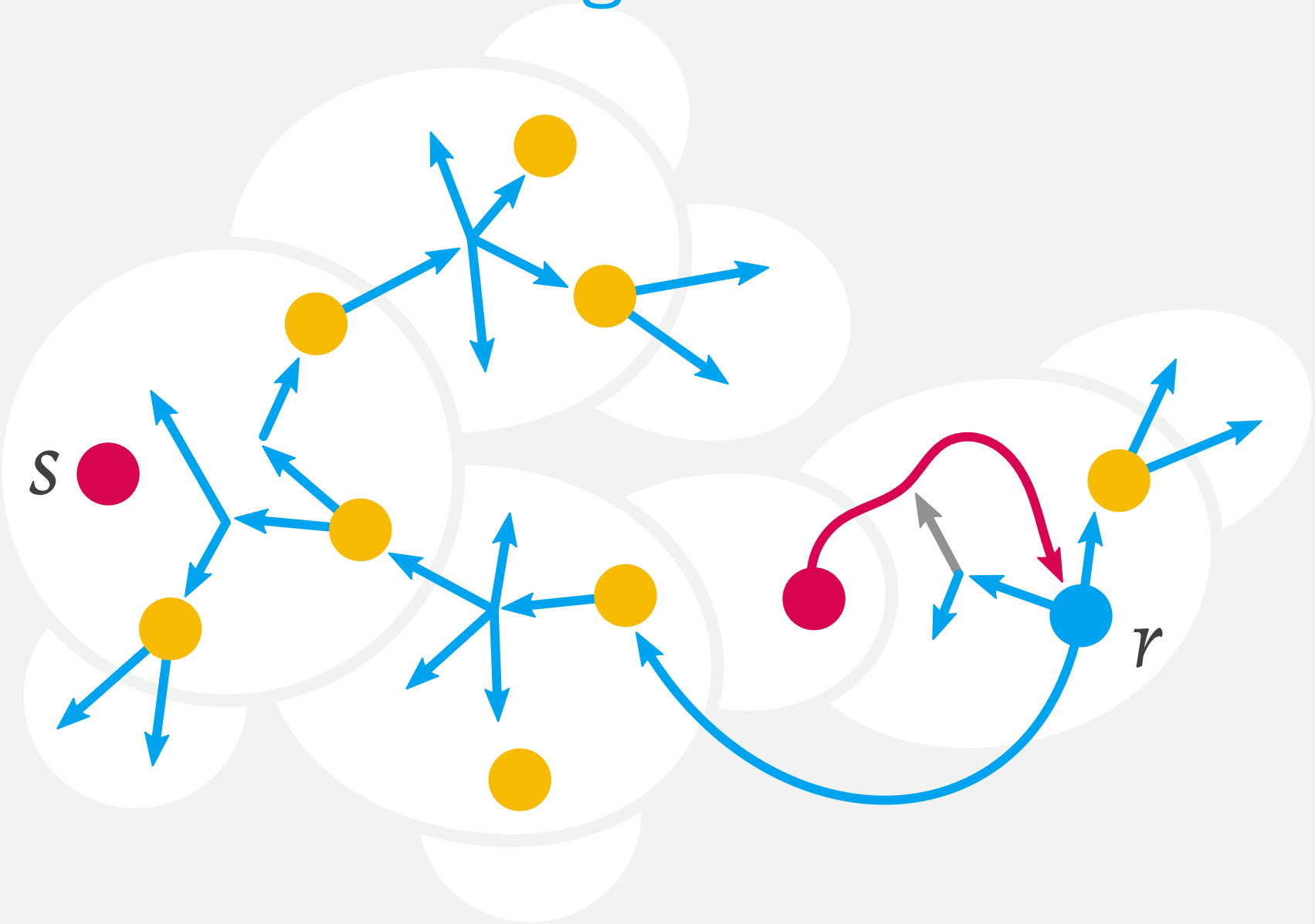

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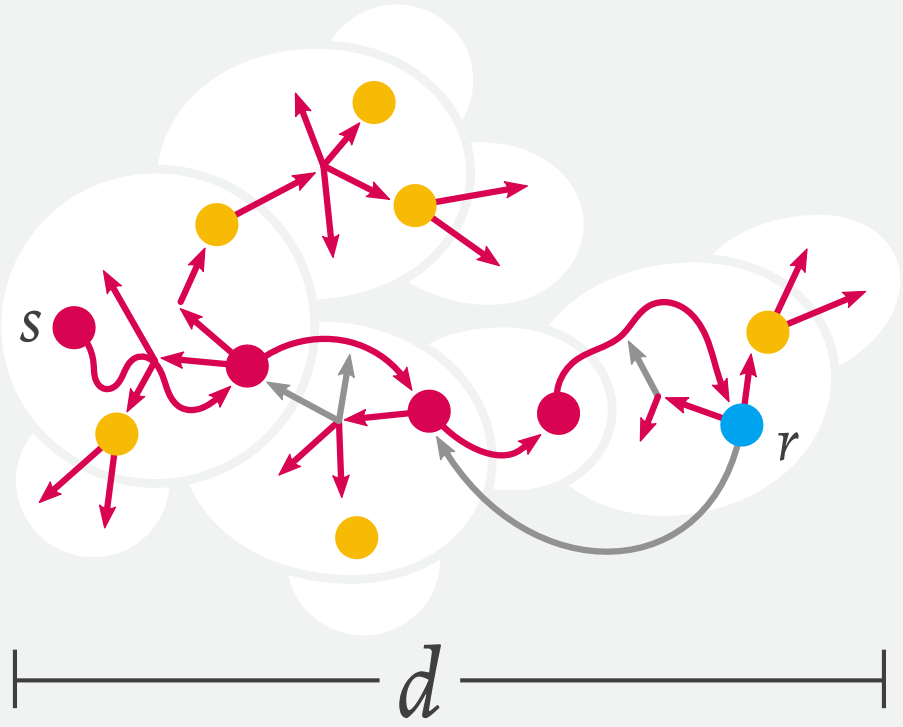
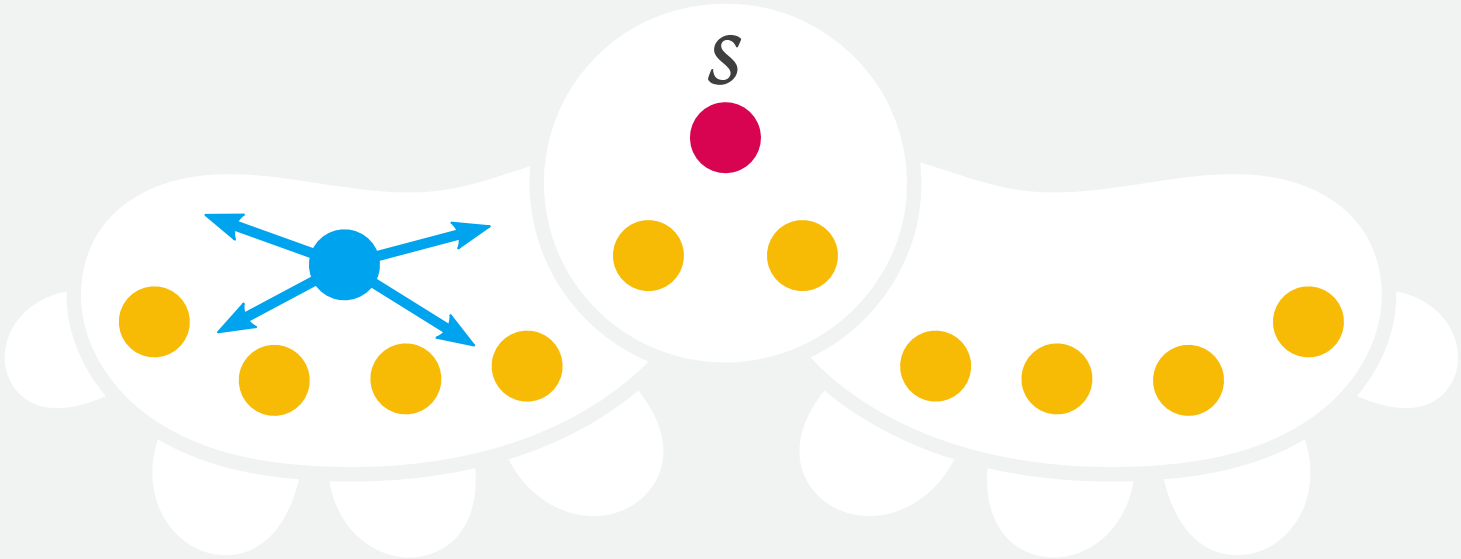


Lemma.
If out-tree had l leaves, then re-rooted tree has $(l\text{-}d)/2$ leaves where d is the **height** of the cut decomposition.

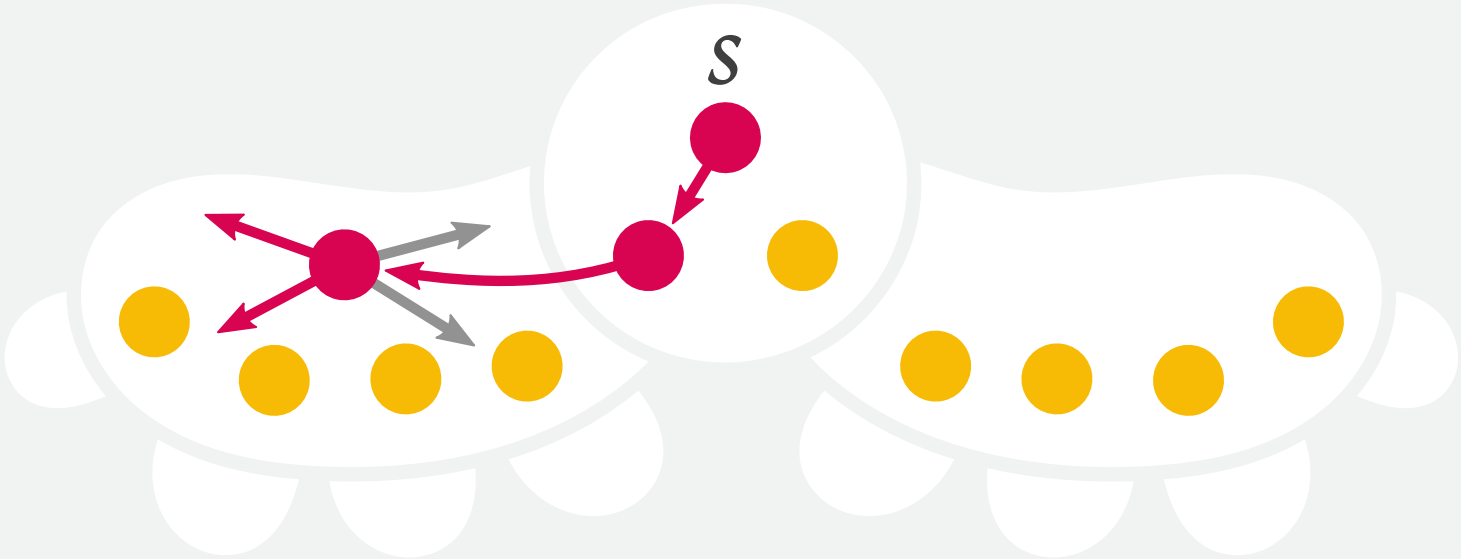
Almost a win-win

If the cut-decomposition is low…



Almost a win-win

If the cut-decomposition is low, we can re-root any out-tree without losing too many leaves



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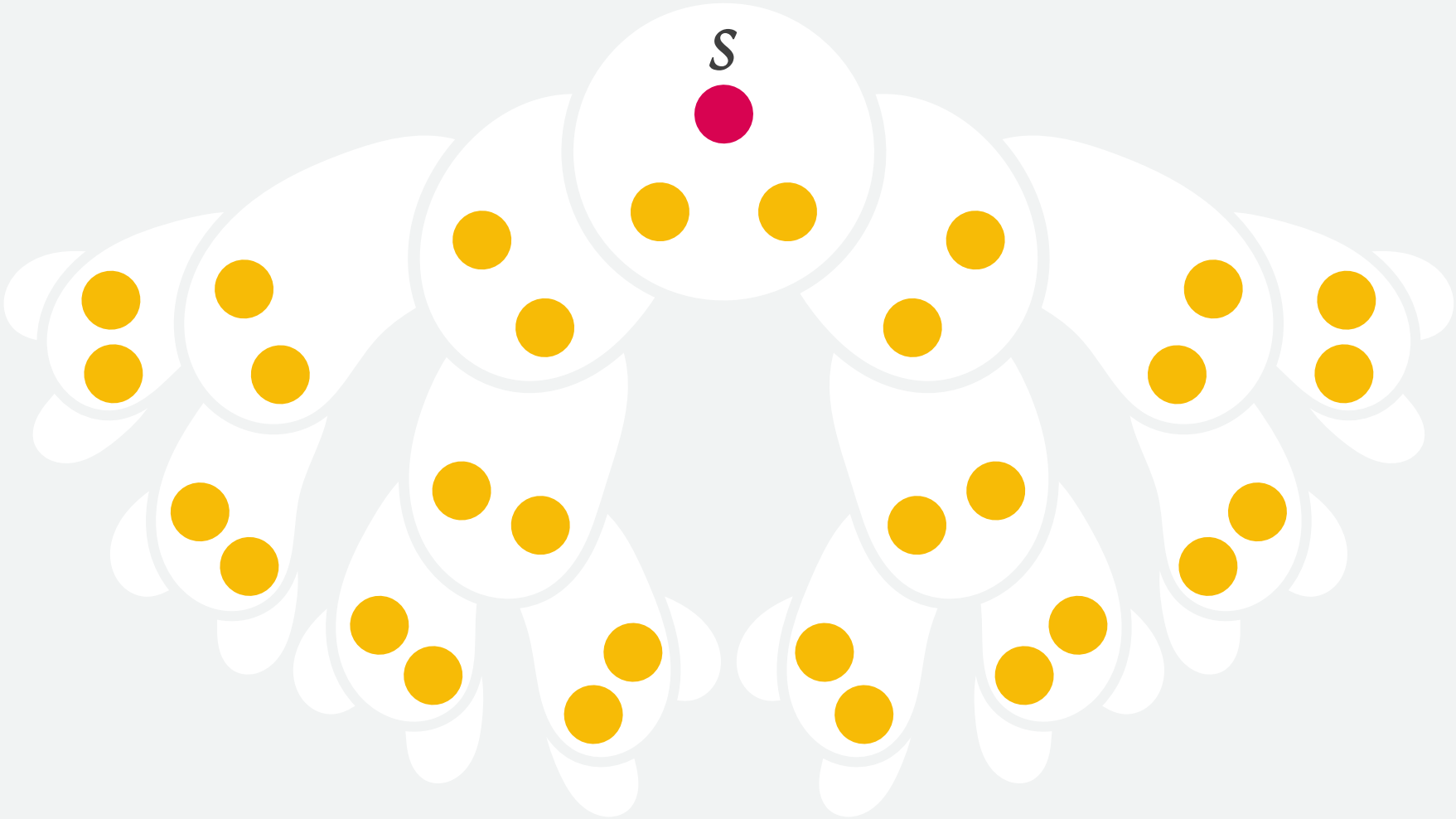
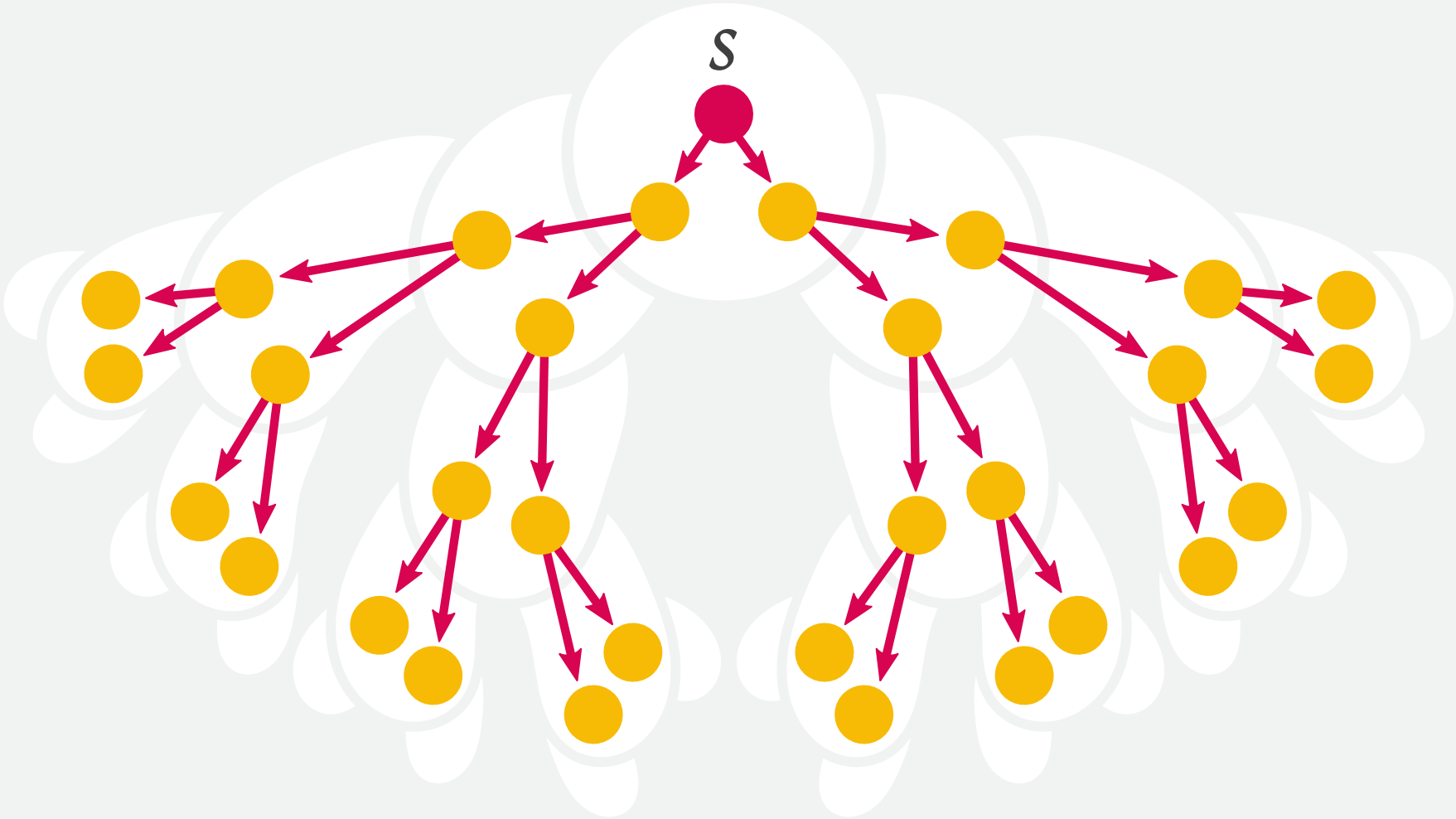
If the cut-decomposition is high...



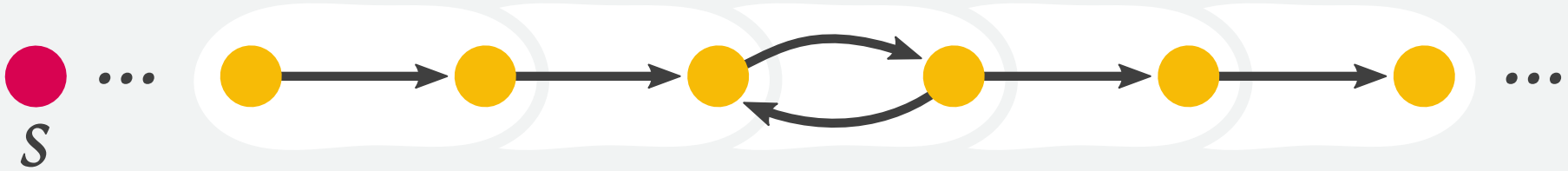
Almost a win-win

If the cut-decomposition is high, we should find an out-tree with many leaves!



Obstacle: degenerate blocks

A diblock of size two is called degenerate.



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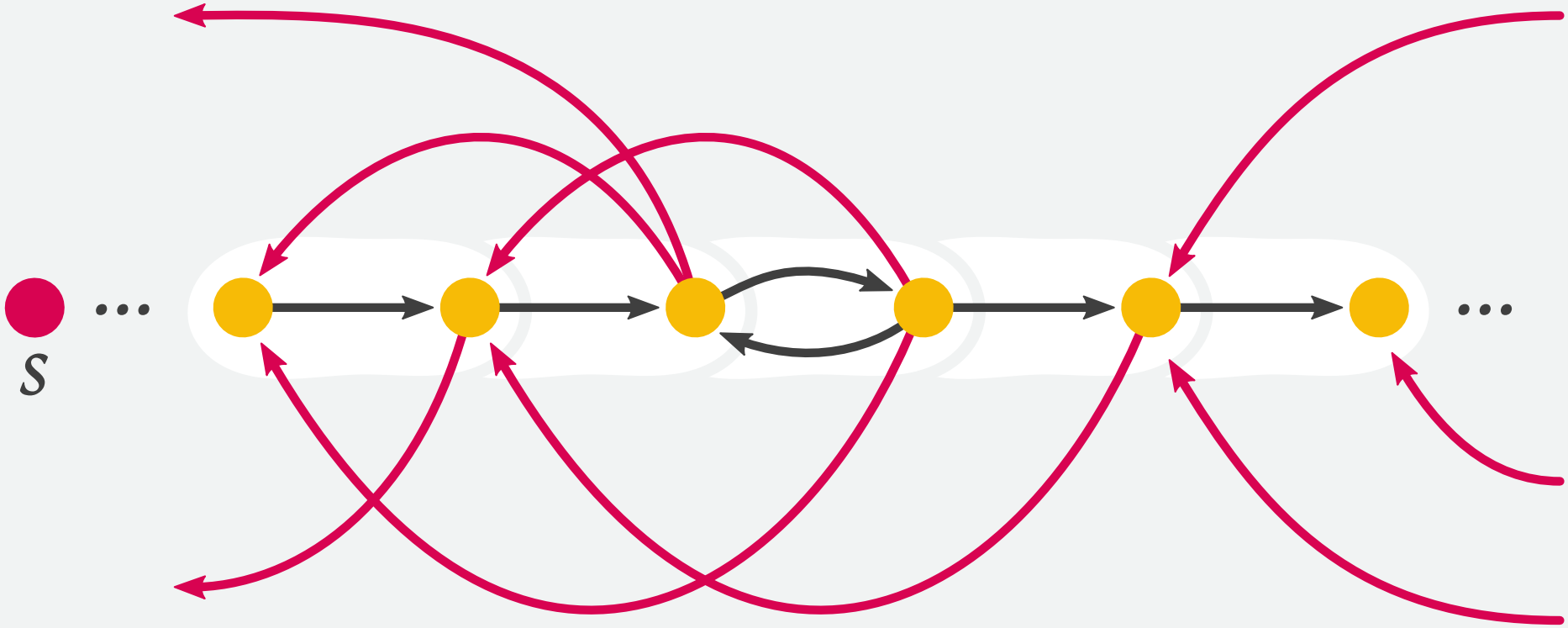

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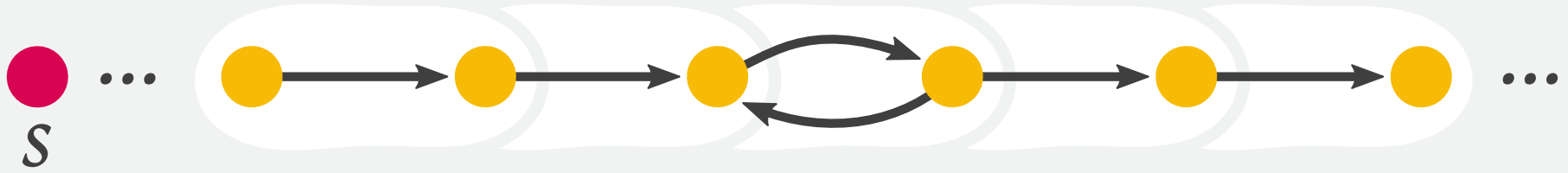

If there exists more than $14k+3$ degenerate diblocks in sequence, we either

1) Apply one of three reduction rules, or
2) construct an in-tree that avoids many arcs.

Schematic of our result (cont'd)

- If D has a rooted out-tree with many leaves, it is a YES-instance
- If D has **no** out-tree with many leaves, it has bounded pathwidth

- If cut decomposition is high: either find a rooted out-tree with many leaves or reduce
- If cut decomposition is low: re-root out-tree with many leaves, keeping many leaves

Summary

- **We show that** SINGLE ROOT k-DISTINCT BRANCHINGS**,** ROOTED k-DISTINCT BRANCHINGS **&** k-DISTINCT BRANCHINGS **are** FPT **in general digraphs**
- **Assuming that a** $2^{O(\mathbf{pw}\ log\ \mathbf{pw})}$ **algorithm exists, our algorithm runs in time** $2^{O(k^2 log^2 k)} n^{O(1)}$

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- $2^{O(k\ log\ k)} n^{O(1)}$ **with more careful analysis..?**
- **Other applications for cut decomposition!**
- **Generalise to larger cut size!**
- **Faster in special graph classes? Cf. sensor network application!**

THANKS!

Questions?