Treewidth and its Characterizations

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1 Introduction

This paper gives a short introduction to the topic of treewidth and focuses on getting a better understanding based on its characterisations. For this the concept of tree decomposition is explained, which is necessary for treewidth and bounded treewidth. There will be given many classes of graphs and shown that they have bounded treewidth of some small constant k. This can be used to solve many NP-complete problems. As an example the maximum independent set problem will be discussed.
Because trees are in generel algorithmically easy to deal with, graph classes with similar properties are sought. The class of bounded treewidth graphs turns out to be such.

2 Preliminaries

At first we will establish some terminology. In the following paper a graph is an ordered pair $G = (V, E)$ containing a set of vertices V and a set of edges E, where $E \subseteq V \times V$ and E is symmetric. Hence all graphs in this paper are undirected. Self-loops and parallel edges dont cause any problems and therefore they are allowed.

Definition:
A tree decomposition of a grah $G = (V, E)$ is a pair $(\{X_i | i \in I\}, T = (I, F))$ where T is a tree with nodes $X_i, X_i \subseteq V$ for all $i \in I$ satisfying

- $\bigcup\limits_{i \in I} X_i = V$
- For all edges $(v, w) \in E$ there exists an $i \in I$ with $v \in X_i, w \in X_i$
- For all $i, j, k \in I$ it holds that if j is on the path from i to k in T then $X_i \cap X_k \subseteq X_j$

The treewidth of a tree decomposition $(\{X_i | i \in I\}, T = (I, F))$ is defined to be $\max\limits_{i \in I} |X_i| - 1$ and the treewidth of a graph G is the minimum treewidth over all tree decompositions of G.
Figure 1 shows an example graph with tree decomposition.

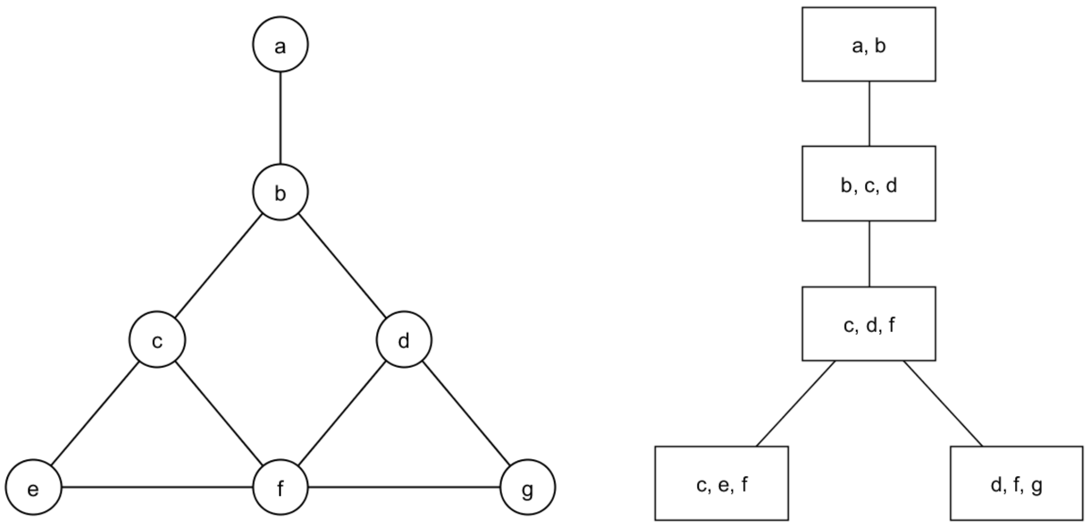

Figure 1: Example graph together with an optimal tree decomposition with treewidth 2

3 Bounded treewidth graphs

Definition:
A class of graphs has a bounded treewidth of k, if the maximum treewidth of all graphs in the class is lower or equal k.

Graphs with bounded treewidth:

- Trees anf forests (treewidth 1)
- Circles (treewidth 2)
- Outerplanar graphs (treewidth 2)
- Series-parallel graphs (treewidths 2)
- Halin graphs (treewidth 3)
- k-outerplanar graphs (treewidth $3^k - 1$)

3.1 Trees and Forests

As mentioned above trees and forests have a bounded treewidth of 1. If so, it holds, that in an optimal tree decomposition $(\{X_i | i \in I\}, T = (I, F))$ of a graph $G = (V, E)$, $|X_i| \leq 2$, for every set X_i. It follows, that for every edge $(v, w) \in V$ there needs to be a set X_i, which contains only v and w. Furthermore a graph G has treewidth 1, if and only if G is a tree or forest. This observation will be proofed later, when we look at circles.

Proof:
Let $G = (V, E)$ be an arbitrary tree. We construct a tree decomposition of G in the following way. Start with a random vertex $v \in V$, $X_0 = \{v\}$, $I = \{0\}$ and iterate over G in depth first search order. Let $w \in V$ be the vertex discovered in the current iteration. Now there are 2 possibilities:

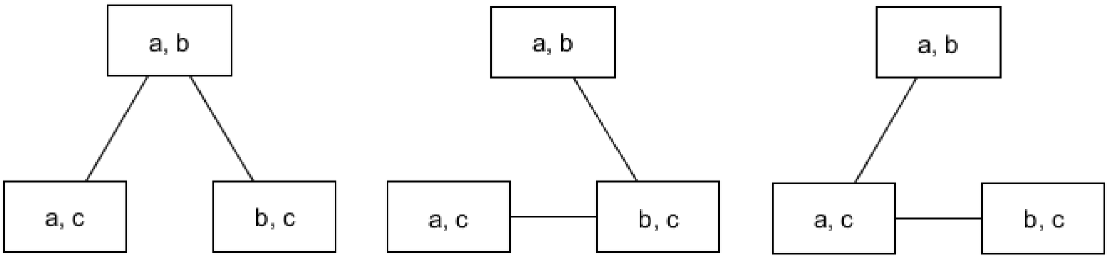

Figure 2: 3 possible attempts to build a tree decomposition of treewidth 2 for the K_3

1 There is an edge $(x, w) \in E$ with $x \in \bigcup\limits_{i \in I} X_i$. Because w was just currently discoverd it holds, that $w \notin \bigcup\limits_{i \in I} X_i$. Now let $x \in X_i$, $i' \notin I$, $I' = I \cup \{i'\}$, $X_{i'} = \{x, w\}$ and $T' = (I', F \cup \{i, i'\})$. Assuming with induction, that we had a tree decomposition before, we now added a new set with one *known* vertex x and one *unknown* vertex w. One can easyly see, that the first two properties are fullfilled. The *known* vertex causes no problems, because the set $X_{i'}$ is connected to a set X_i, which contains x. So for X_i the third property was already fullfilled. The only paths that are new in $(\{X_i | i \in I'\}, T')$ lead over X_i, so the property is fullfilled. The *unknown* vertex w causes no problems, because there cannot be two sets $X_j, X_k \in \{X_i | i \in I\}$ with $X_j \cap X_k = \{w\}$. So $(\{X_i | i \in I'\}, T')$ is a tree decomposition of the graph induced by the already discovered vertices.

2 There is no edge $(x, w) \in E$ with $x \in \bigcup\limits_{i \in I} X_i$. Then G is a forest and we can proceed with the next vertex. We just need to connect the components of the tree decomposition randomly, because there are only different vertices in the components. $\square$

3.2 Circles

Circles have a bounded treewidth of 2. At first we will see, why circles cannot have treewidth 1 and proof, that if and only if a graph has treewidth 1 it is a tree or forest. For this we will take a look at the circle with 3 vertices and 3 edges, $G = (V = \{a, b, c\}, E = \{(a, b), (a, c), (b, c)\})$. If we want to consruct a tree decomposition with treewidth 1, we have one set for every edge again, $X_0 = \{a, b\}$, $X_1 = \{a, c\}$, $X_2 = \{b, c\}$. Now there are 3 possible ways of building a tree with 3 vertices. For each $i \in I$ there is a tree, where i is between the two other vertices, as its shown in figure 2. $X_0 \cap X_1 = \{a\} \subsetneq X_2$, $X_0 \cap X_2 = \{b\} \subsetneq X_1$, $X_1 \cap X_2 = \{c\} \subsetneq X_0$. So its not possible to fullfill the third property with sets with 2 elements. The same method can be used analog for arbitrary circles.
Next we will proof, that circles have a treewidth of 2. For an easier understanding we imagine a circle as two paths, which are connected at their ends.

Proof:
Let $G = (V, E)$ be a circle. Let $G_1 = (V_1, E_1)$ be the subgraph of G induced by a path with length $|V|/2$ and $G_2 = (V_2, E_2)$ the subgraph induced by $V \setminus V_1$. Further $a_1 \neq b_1 \in V_1$ be the vertices with degree 1 in G_1 and $a_2 \neq b_2 \in V_2$ the vertices with degree 1 in G_2, so that $(a_1, a_2), (b_1, b_2) \in E$. For both of these subgraphs one can construct a tree de-

composition with treewidth 1. For a graph $G_3 = (V_1 \cup V_2, E_1 \cup E_2 \cup \{(a_1, a_2)\})$ its still possible by connecting the tree decompostions of G_1 and G_2 in the right way. If we now add the edge (b_1, b_2) and connect b_1, b_2 to one end of the tree decomposition of $G3$, the third property is not fullfilled anymore. But the only proplematic case is the patch from one end of hte tree decomposition to the other, where either b_1 or b_2 is in the cut. Assuming b_1 is in the cut we can now modify each set X_i, by adding b_1. By this we obtain a tree decomposition of G with treewidth 2. $\square$

3.3 Complete Graphs

To show that complete graphs with n vertices have a treewidth of $n-1$ we take a look at the graph with 4 vertices. We will see, that for complete graphs the tree decomposition consists of only one set which contains all vertices. The given proof can be transferred to any complete graph.

Proof:
Let $G = (V = \{a, b, c, d\}, E)$ be the complete graph with 4 vertices. To obtain a treewidth smaller than $n-1$ we need to build sets with less than n elements. Without limitation of generality we assume the sets $\{a, b, c\}, \{b, c, d\}$ to be taken. To cover all edges we need to add the set $\{a, d\}$. We can now easily see that the path condition of tree decompositions cant be fullfilled anymore. Because $\{a, b, c\} \cap \{a, d\} = \{a\} \subsetneq \{b, c, d\}$, $\{b, c, d\} \cap \{a, d\} = \{d\} \subsetneq \{a, b, c\}$, $\{a, b, c\} \cap \{b, c, d\} = \{b, c\} \subsetneq \{a, d\}$. $\square$

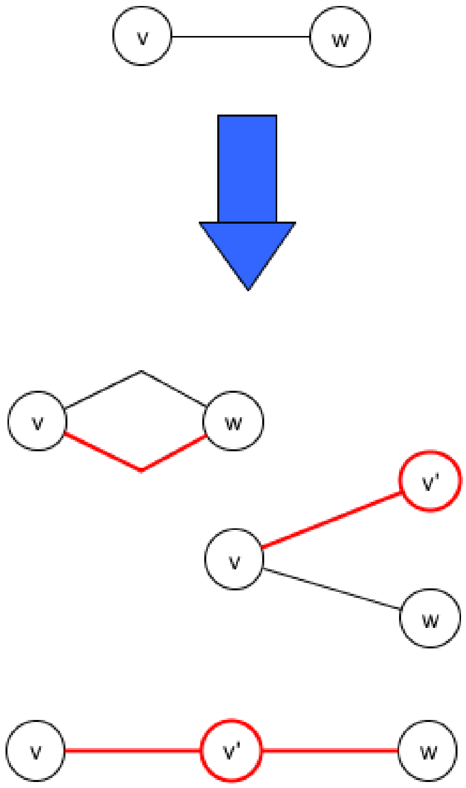

Figure 3: 3 operations that can be used to obtain a series-parallel graph

3.4 Series-parallel Graphs

Definition:
The graph with 2 vertices and one edge is series-parallel. Let $G = (V, E)$ be a series-parallel graph. Another series-parallel graph G' can be obtained by one of the following 3 ways, also shown in figure 3:

1 replacing an edge by two parallel edges

2 adding a new vertex v' and a new edge (v, v') with $v \in V$

3 adding a new vertex v' and replacing an edge $(v, w) \in E$ by the new edges (v, v') and (v', w)

Series-parallel graphs have a bounded treewidth of 2.

Proof:
Let G be a series-parallel graph and $(\{X_i | i \in I\}, T = (I, F))$ a tree decomposition of G with treewidth smaller or equal 2 and G' a graph, which is obtained by one of the 3 operations mentioned in the definition, ergo a series-parallel graph. We can assume this, because we can always start with the graph with 2 vertices and one edge, which obviously has treewidth 1. Given that, we can build a tree decomposition of G' as follows:

- if operation 1 is used, the tree decomposition of G already is a tree decomposition of G'
- if operation 2 is used, a new vertex v' and a new edge (v, v') is added. Now let $i' \notin I$, and $X_{i'} = \{v, v'\}$. Because v is not a new vertex there must be X_i with $v \in X_i$. Furthermore let $I \cup \{i'\}$ and $T' = (I', F \cup \{(i, i')\})$.
- if operation 3 is used, an edge (v, w) is replaced and a new vertex v' is added. If an edge (v, w) is replaced, there must be $i \in I$ with $v, w \in X_i$. Now let $i' \notin I$, $I' = T \cup i'$, $X_{i'} = \{v, w, v'\}$ and $T' = (I', F \cup \{i, i'\})$.

Than $(\{X_i | i \in I'\}, T')$ is a tree decomposition of G' with treewidth$(G') \leq 2$. The treewidth cannot grow larger than 2, because we add at maximum a set with 3 Elements. Also the added sets fullfill the first two properties of a tree decomposition. The remaining path condition is fullfilled too, because for the already *known* vertices in the added set the path condition was fullfilled in the old tree decomposition and we connected it to a set which contained the *known* vertices already. The *unknown* vertex v' cannot violate the condition, because it occurres only one time in the tree decomposition. So there is no possible vertex of the tree decomposition with $X_{i'} \cap X_i = v'$. $\square$

Next we take a look at outerplanar graphs. A graph is outerplanar, if one can add a new vertex which is connected to all other vertices and the graph remains planar. It can be shown, that every outerplanar graph is series-parallel. Because of that outerplanar graphs have a bounded treewidth of 2. Furthermore it can be shown, that every graph with treewidth 2 or less is series parallel.
K-outerplanar graphs have a bounded treewidth of $3^k - 1$. The proof can be found in [4].

3.5 Planar Graphs

In general planar graphs do not have bounded treewidth. The class of grid graphs are planar and it can be shown, that the $n \times n$ grid graph has treewidth of exactly n. In figure 4 the 3×3 grid graph is shown, together with a tree decomposition of treewidth 3.

4 Solving Independent Set with Tree Decomposition

We now take a look at the maximum independent set problem, which is a well known NP-hard problem. With the help of the tree decomposition of a graph we will show an algorithm, which determines the maximum independent set in polynomial time.

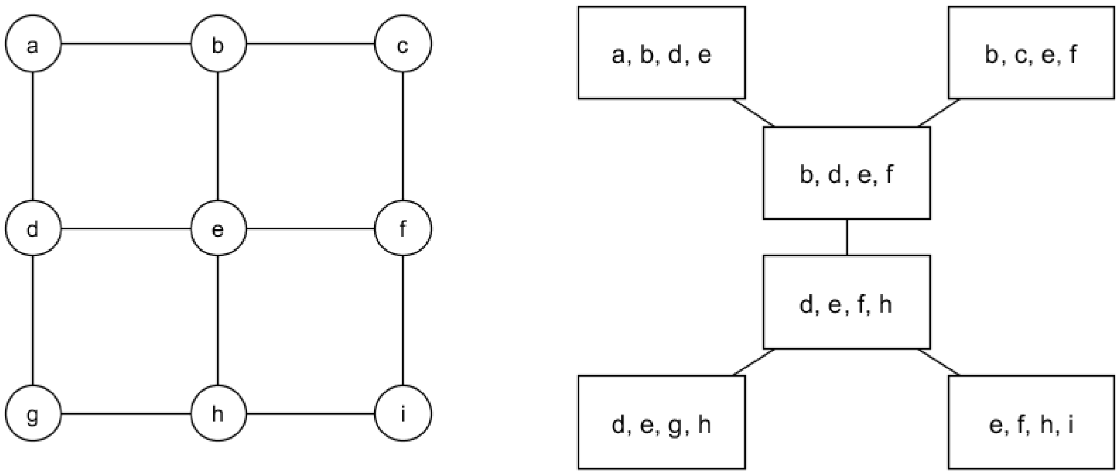

Figure 4: 3×3 grid graph and an optimal tree decomposition of treewidth 3

Definition:
A maximum independent set of a graph $G = (V, E)$ is a subset $W \subseteq V$ with maximum size, so that for all $v, w \in W$ it holds that $(v, w) \notin E$.

Let $(\{X_i | i \in I\}, T = (I, F))$ be a tree decomposition of a graph G with treewidth k. Then there is always a rooted binary tree decomposition of G with treewidth k. This means a tree decomposition, with a fixed root and every vertex which is no leave has 2 childs. For a rooted binary tree decomposition of G let $G[X_i]$ be the graph induced by X_i and all of his childs on G.
We can now use a dynamic program to solve the problem in the following way:
We assume $Z = W \cap X_i$ is the independent subset in $G[X_i]$ and $s_i(Z)$ the size of this subset. For leave nodes X_i all $2^{|X_i|}$ values of $s_i(Z)$ are found by:

$$s_i(Z) = \begin{cases} |Z|, & \text{if } \forall v, w \in Z : (v, w) \notin E \\ -\infty, & \text{if } \exists v, w \in Z : (v, w) \in E \end{cases}$$

For inner nodes i and child nodes j and k and if for all $v, w \in Z : (v, w) \notin E$ the formula is:

$$s_i(Z) = \max_{Z \cap X_j = Z' \cap X_i, Z \cap X_k = Z'' \cap X_i} \{s_j(Z') + s_k(Z'') + |Z \cap (X_i \setminus X_j \setminus X_k)| - |Z \cap X_j \cap X_k|\}$$

and $s_i(Z) = -\infty$ for $v, w \in Z : (v, w) \in E$. Z' and Z'' denote the Z for the left and right child of i.
The given algorithm generates a maximum independent set for a graph G if given a tree decomposition of G with treewidth k in time $\mathcal{O}(n \cdot 2^{3k})$.

5 References

[1] A. Krause. Bounded Treewidth Graphs - A Survey, German Russian Winter School, St. Petersburg, Russia. 2003.

[2] H. L. Bodlaender. A Tourist Guide through Treewidth. 1993.

[3] H. L. Bodlaender. Planar Graphs with Bounded Treewidth. 1988.

[4] H. L. Bodlaender. Classes of Graphs with Bounded Treewidth. 1986.