Problem Kernels

Let L be a parameterized problem.

Sometimes you can answer the question $(w, k) \in L$ as follows:

- If k is very big, use brute force.
- If k is small and w is complicated, then (w, k) cannot be a solution.
- If k is small and w is simple, then we can easily solve $(w, k) \in L$.

Problem Kernels

Definition

A function $f\colon \Sigma^* \times \mathbf{N} \to \Sigma^* \times \mathbf{N}$ is a reduction to a problem kernel for a parameterized problem L, if

- $(w, k) \in L$ iff $f(w, k) \in L$,
- there is a function $f'\colon \mathbf{N} \to \mathbf{N}$, such that $|w'| \leq f'(k)$, if $f(w, k) = (w', k')$,
- f can be computed in polynomial time.

In a nutshell: A reduction to a problem whose size is limited by a function of the parameter.

Example Vertex Cover

Assume some graph has a vertex cover of size k.

Let v be a vertex whose degree is at least $k+1$.

Question:

Must v belong to the vertex cover of size k?

Reduction to a problem kernel:

If there is a node with degree $> k$, remove it. The original graph has a VC of size k iff the reduced graph has a VC of size $k-1$.

Example Vertex Cover

Assume some graph has a vertex cover of size k.

Let v be a vertex whose degree is at least $k+1$.

Question:

Must v belong to the vertex cover of size k?

Reduction to a problem kernel:

If there is a node with degree $> k$, remove it. The original graph has a VC of size k iff the reduced graph has a VC of size $k-1$.

Example Vertex Cover

Question:

How big is the resulting graph at most?

(if we also remove isolated vertices)

Answer:

- The vertex cover itself consists of only k nodes.
- Each of these k nodes can have at most k neighbors.
- There can be at most $k(k+1)$ nodes in total.

Example Vertex Cover

Question:

How big is the resulting graph at most?

(if we also remove isolated vertices)

Answer:

- The vertex cover itself consists of only k nodes.
- Each of these k nodes can have at most k neighbors.
- There can be at most $k(k+1)$ nodes in total.

A smaller Problem Kernel

Theorem (Nemhauser and Trotter)

Let $G = (V, E)$ be a graph of n nodes and m edges.
It takes only polynomial time to find two disjoint node sets C_0 and V_0 such that

1. *If $D \subseteq V_0$ is a vertex cover of $G[V_0]$, then $D \cup C_0$ is a vertex cover of G.*
2. *There is an optimal vertex cover of G containing all of C_0.*
3. *Every vertex cover of $G[V_0]$ has size at least $|V_0|/2$.*

A smaller Problem Kernel

Theorem (Nemhauser and Trotter)

Let $G = (V, E)$ be a graph of n nodes and m edges.
It takes only polynomial time to find two disjoint node sets C_0 and V_0 such that

1. *If $D \subseteq V_0$ is a ... rtex cover of G.*
2. *There is an optimal vertex cover of G containing all of C_0.*
3. *Every vertex cover of $G[V_0]$ has size at least $|V_0|/2$.*

If G has a vertex cover of size k, then
$|V_0| + |C_0| \leq 2k$
Why???

A smaller Problem Kernel

Theorem (Nemhauser and Trotter)

Let $G = (V, E)$ be a graph of n nodes and m edges.
It takes only polynomial time to find two disjoint node sets C_0 and V_0 such that

1. *If $D \subseteq$ … cover of G.*
2. *There is an optimal vertex cover of G containing all of C_0.*
3. *Every vertex cover of $G[V_0]$ has size at least $|V_0|/2$.*

An optimal vertex cover of $G[V_0]$ combined with C_0 is an optimal vertex cover of G.
Why???

Why???

A smaller Problem Kernel

This results in the following algorithm that reduces (G, k) to $(G[V_0], k')$.

- Compute C_0 and V_0
- Let $k' = k - |C_0|$
- G now has a vertex cover of size k if and only if $G[V_0]$ has a vertex cover of size k'.

If $2k' < |V_0|$, then G cannot have a vertex cover of size k.

A smaller Problem Kernel

The following algorithm solves Vertex Cover:

1. Compute V_0 and C_0
2. Output No if $2(k - |C_0|) < |V_0|$
3. Compute an optimal vertex cover C_1 of $G[V_0]$
4. If $|C_1| + |C_0| \leq k$ output Yes and No otherwise

Running time: $n^{O(1)} + O(k2^k)$

Proof of the Nemhauser–Trotter Theorem

An algorithm that computes C_0 and V_0:

Let $G = (V, E)$, V' be a disjoint copy of V, and $G_B = (V, V', E_B)$ be the bipartite subgraph such that

$$\{x, y'\} \in E_B \iff \{x, y\} \in E.$$

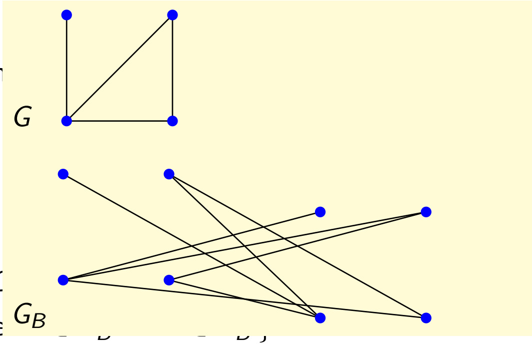
- Compute an optimal vertex cover C_B for G_B.
- Let $C_0 = \{\, x \mid x \in C_B \text{ and } x' \in C_B \,\}$.
- Let $V_0 = \{\, x \mid \text{ either } x \in C_B \text{ or } x' \in C_B \,\}$.

Proof of the Nemhauser–Trotter Theorem

An algorithm that computes C_0 and V_0:

Let $G = (V, E)$, V' be a
be the bipartite subgraph

$\{x, y'\}$



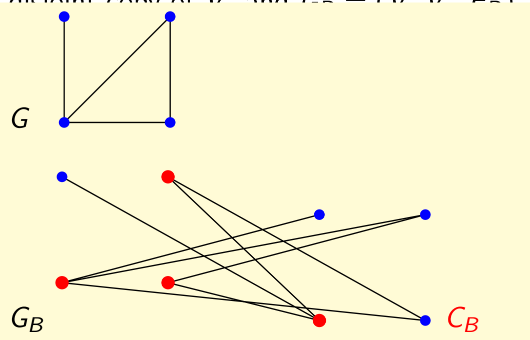
- Compute an optima
- Let $C_0 = \{\, x \mid x \in C$
- Let $V_0 = \{\, x \mid$ eithe

Proof of the Nemhauser–Trotter Theorem

An algorithm that computes C_0 and V_0:

Let $G = (V, E)$, V' be a
be the bipartite subgraph

$\{x, y'\}$



- Compute an optima
- Let $C_0 = \{ x \mid x \in C$
- Let $V_0 = \{ x \mid$ eithe

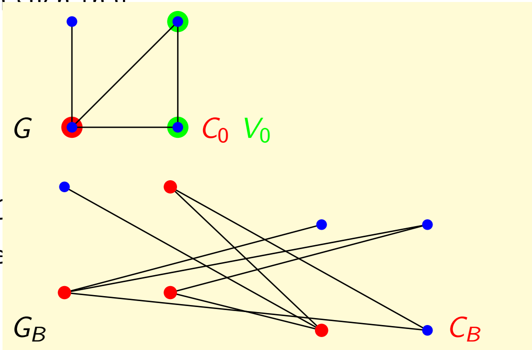
Proof of the Nemhauser–Trotter Theorem

An algorithm that computes C_0 and V_0:

Let $G = (V, E)$, V' be a disjoint copy of V, and $G_B = (V, V', E_B)$ be the bipartite subgraph such that

$$\{x, y'\}$$

- Compute an optima
- Let $C_0 = \{ x \mid x \in C$
- Let $V_0 = \{ x \mid$ eithe



Proof of the Nemhauser–Trotter Theorem

Obviously,

- C_0 and V_0 are disjoint
- C_0 and V_0 can be computed in polynomial time

We need to prove the three statements of the theorem:

1. If $D \subseteq V_0$ is a vertex cover of $G[V_0]$, then $D \cup C_0$ is a vertex cover of G.
2. There is an optimal vertex cover of G containing all of C_0.
3. Every vertex cover of $G[V_0]$ has size at least $|V_0|/2$.

Statement 1

Claim: If $D \subseteq V_0$ is a vertex cover of $G[V_0]$, then $D \cup C_0$ is a vertex cover of G.

Let $D \subseteq V_0$ be a vertex cover of $G[V_0]$ and $e = \{x, y\} \in E$ an arbitrary edge.

Let $I_0 = V - V_0 - C_0$.

- If an endpoint of e is in C_0... okay
- If both endpoints are in V_0... okay
- $x \in I_0 \;\Rightarrow\; y, y' \in C_B \;\Rightarrow\; y \in C_0, \ldots$ okay

Statement 2

Claim: There is an optimal vertex cover of G containing all of C_0.

Let S be an optimal vertex cover and $S_V = S \cap V_0$, $S_C = S \cap C_0$, $S_I = S \cap I_0$, $\bar{S}_I = I_0 - S_I$.

Lemma

$(V - \bar{S}_I) \cup S'_C$ is a vertex cover of G_B.

Proof

Let $\{x, y'\} \in E_B$.

If $x \notin \bar{S}_I$, then $x \in (V - \bar{S}_I) \cup S'_C$.

If $x \in \bar{S}_I$, then $x \in I_0$, $x \notin S \Rightarrow y \in S$, $y, y' \in C_B \Rightarrow$
$\Rightarrow y \in C_0 \Rightarrow y \in S \cap C_0 = S_C \Rightarrow y' \in S'_C$.