Statement 2

$$
\begin{aligned}
|V_0| + 2|C_0| &= |V_0 \cup C_0 \cup C_0'| \\
&= |C_B| \\
&\leq |(V - \bar{S}_I) \cup S_C'| \text{ due to the lemma} \\
&= |V - \bar{S}_I| + |S_C'| \\
&= |V_0 \cup C_0 \cup I_0 - (I_0 - S_I)| + |S_C'| \\
&= |V_0| + |C_0| + |S_I| + |S_C|
\end{aligned}
$$

It follows that $|C_0| \leq |S_I| + |S_C| = |S| - |S_V|$ and thus $|C_0 \cup S_V| \leq |S|$.

Statement 3

Claim: Every vertex cover of $G[V_0]$ has size at least $|V_0|/2$.

Let S_0 be an optimal vertex cover of $G[V_0]$.

$C_0 \cup C_0' \cup S_0 \cup S_0'$ is a vertex cover of G_B, because $C_0 \cup S_0$ is a vertex cover of G.

$$|V_0| + 2|C_0| = |C_B| \leq |C_0 \cup C_0' \cup S_0 \cup S_0'| = 2|C_0| + 2|S_0|$$

The claim follows.

Graph Properties

Definition

A graph property Π is a class of graphs that is closed under graph isomorphisms.

That is, if two graphs G_1 and G_2 are isomorphic, both belong to Π or both don't.

Graph Properties

Example

- Connected graphs
- Trees
- Graphs containing a clique of size 100
- Planar graphs
- Regular graphs
- Finite graphs

Graph Properties

These are not graph properties:

- Graphs whose nodes are natural numbers
- Every nonempty finite set of graphs
- (For Logicians: Each set of graphs)

Hereditary Graph Properties

A graph property Π is called hereditary if the following holds:

Let $G \in \Pi$ and H be an induced subgraph of G.

Then $H \in \Pi$ as well.

In other words: Π is closed under taking induced subgraphs.

Hereditary Graph Properties

A graph property Π is called hereditary if the following holds:

Let $G \in \Pi$ and H be an induced subgraph of G.

Then $H \in \Pi$ as well.

In other wor

Questions:

1. Does the empty graph belong to every hereditary graph property?
2. Are graph properties lattices with respect to the induced subgraph relation?

Hereditary Graph Properties

Which graph properties are hereditary?

- Bipartite graphs
- Complete graphs
- Planar graphs
- Trees
- Connected graphs
- Graphs of diameter at most d
- Regular graphs

Hereditary Graph Properties

Which graph properties are hereditary?

- Forests
- Graphs containing an independent set of size 8
- Graphs with at least 17 nodes
- Graphs containing no matching of size 35
- 5-regular graphs
- Infinite graphs
- Chordal graphs

Characterization by Obstruction Sets

Definition

A graph property Π has a characterization by obstruction sets if there is a graph property $\mathcal{F}$ such that $G \in \Pi$ if and only if $\mathcal{F}$ does not contain an induced subgraph of G.

Characterization by Obstruction Sets

Definition

A graph property Π has a chara if there is a graph property $\mathcal{F}$ suc does not contain an induced subgrap

Question:
Does every hereditary graph property have a characterization by obstruction sets?

Characterization by Obstruction Sets

Definition

A graph property Π has a chara if there is a graph property $\mathcal{F}$ suc does not contain an induced subgrap

Question:
Does every hereditary graph property have a

Answer:
Yes. Choose $\mathcal{F} = \mathcal{G} - \Pi$ with $\mathcal{G}$ containing all graphs.

Finite Obstruction Sets

Definition

A graph property Π has a finite characterization by obstruction sets if it has a characterization by $\mathcal{F}$, and $\mathcal{F}$ contains only a finite number of non-isomorphic graphs.

Finite Obstruction Sets

Which graph properties have a finite characterization by obstruction sets?

- Graphs containing an independent set of size 7?
- Bipartite graphs?
- Forests?
- Planar graphs?
- 5-colorable graphs?
- Graphs containing a vertex cover of size k?

Finite Obstruction Sets

Which graph properties have a finite characterization by obstruction sets?

- Triangle-free graphs?
- Graphs without any k-cliques?
- Graphs of diameter at most d?
- Cycle-free graphs?
- Graphs not containing any cycle of length k?

Graph Modification Problems

Let Π be a graph property. There are the following well-known graph modification problems for an input G:

1. Edge Deletion Problem: Can we obtain a graph in Π by deleting k edges from G?
2. Node Deletion Problem: Can we obtain a graph in Π by deleting k nodes from G?
3. Node/Edge Deletion Problem: Can we obtain a graph in Π by deleting k nodes and l edges from G?
4. Edge Insertion Problem: Can we obtain a graph in Π by inserting k edges in G?

Generalization

Definition

$\Pi_{i,j,k}$-Graph Modification Problem

Input: A graph $G = (V, E)$
Parameter: $i, j, k \in \mathbf{N}$
Question: Can we obtain a graph in Π by removing up to i nodes, removing up to j edges, and inserting up to k edges in G?

The Leizhen Cai Theorem

Theorem

Let Π be a graph property with a finite characterization by obstruction sets.
Then the $\Pi_{i,j,k}$-Graph Modification Problem can be solved in $O(N^{i+2j+2k}|G|^{N+1})$ steps and is thus fixed parameter tractable.
N is the number of nodes in the largest graph in the obstruction set, i.e., a constant.

Proof of the Leizhen Cai Theorem

Lemma

Let Π be a hereditary graph property that can be checked in $T(G)$ steps.
Then it takes $O(|V|T(G))$ many steps to find a minimal forbidden induced subgraph for any $G = (V, E) \notin \Pi$.

In this context, the term "minimal" refers to the order "induced subgraph".

Proof of the Leizhen Cai Theorem

Proof of the lemma:

Let $V = \{v_1, \ldots, v_n\}$.

$H := G$
for $i = 1, \ldots, n$ **do**
  if $H - \{v_i\} \notin \Pi$ **then** $H := H - \{v_i\}$
od

Upon termination, H is a minimal forbidden induced subgraph.

Proof of the Leizhen Cai Theorem

Input: $G = (V, E)$

Parameter: $i, j, k \in \mathbf{N}$

Question: $G \in \Pi_{i,j,k}$

while $i + j + k > 0$ **do**
  $H :=$ minimal forbidden induced subgraph of G
  Modify G by removing an edge or node or by inserting an edge from/in H
  Let $i := i - 1$, $j := j - 1$, or $k := k - 1$.
  if $G \in \Pi$ **then** answer YES
od
Answer NO

Proof of the Leizhen Cai Theorem

Running time:

Find H: $O(|V| \cdot |V|^N)$ according to the lemma

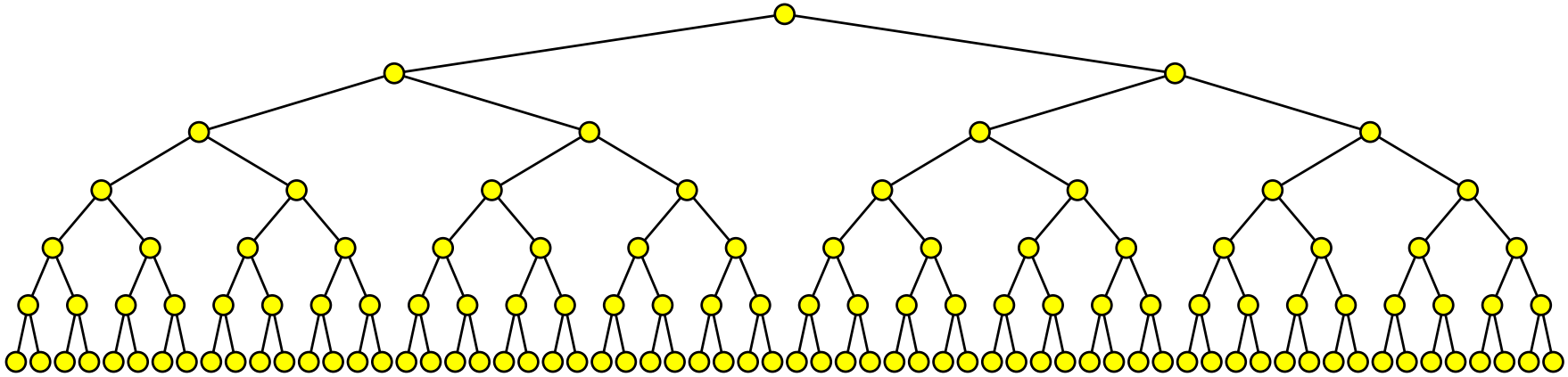
There are only N ways of removing a node from H

There are only $\binom{N}{2}$ ways of deleting or inserting an edge from/in H

Total running time

$$O(N^{i+2j+2k}|V|^{N+1}).$$

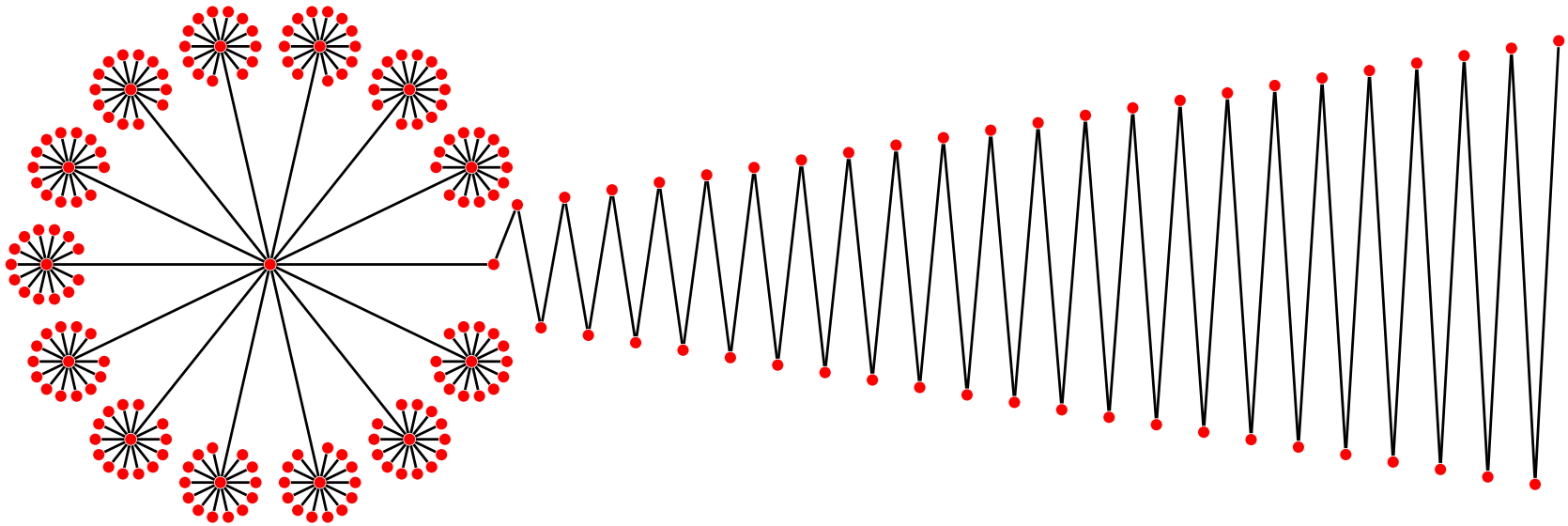
Interleaving — Search Trees and Problem Kernels



In a search tree, the parameter decreases as it approaches the leaves, whereas this is not necessarily the case for the size of the instance (which could even grow).

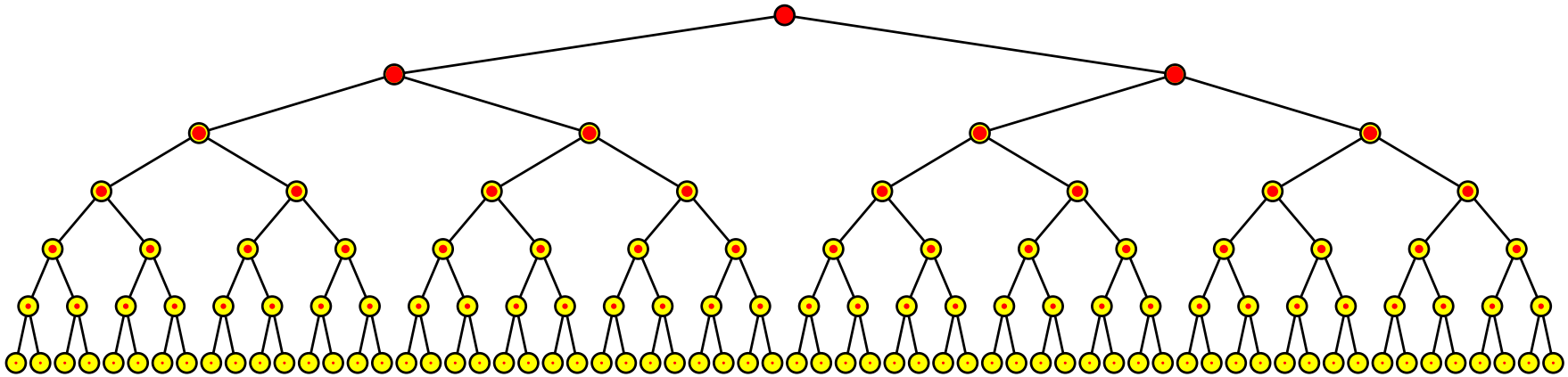
If the expansion of a node in the search tree takes $p(n)$ steps, then the total running time becomes $O(s(k, n)p(n))$, where $s(k, n)$ denotes the size of the search tree.

Interleaving



The size of this graph never drops below $n/2$ within the entire search tree of a vertex cover algorithm, provided that no reduction to the problem kernel is performed.

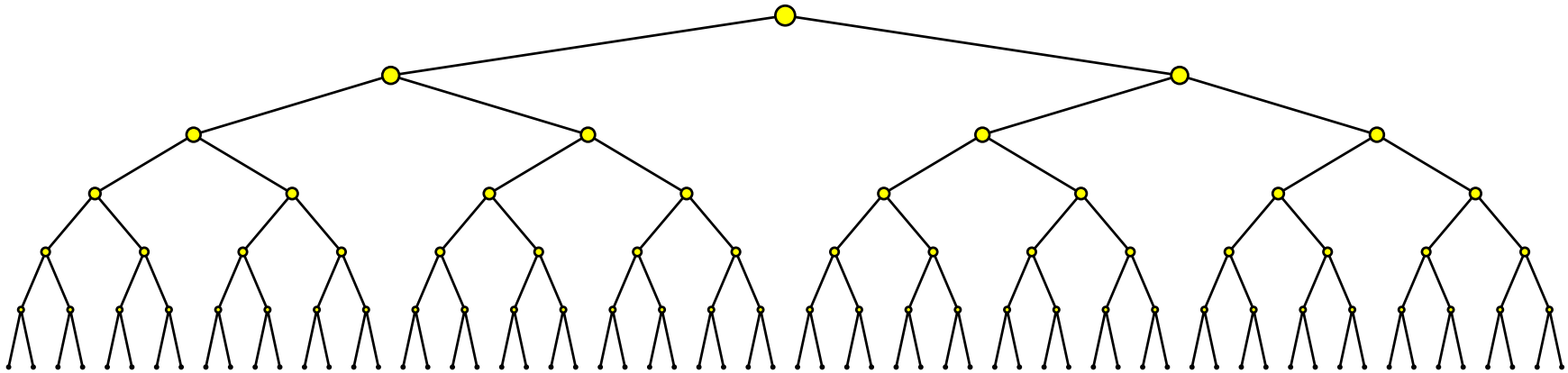
Interleaving



The size of the parameter is reflected by the size of the red dots. The recursion stops when the parameter is small. In the leaves, the parameter is bounded by a constant.

Nearly all nodes are close to a leaf $\Longrightarrow$ nearly all nodes have a small parameter.

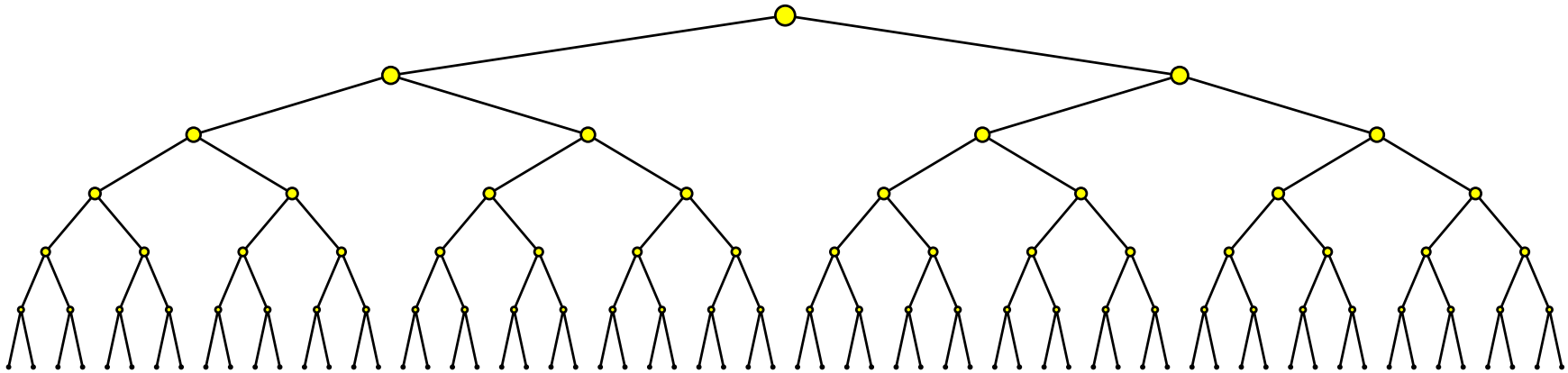
Interleaving



If we perform a reduction to problem kernel after each expansion of a node in the search tree, then the instances decrease in size as we approach the leaves.

A more detailed analysis reveals that the total running time is only $O(s(n,k) + t(n) + r(n))$ rather than $O(s(n,k)t(n))$, where $r(n)$ denotes the time required for the reduction to problem kernel.

Search Trees and Dynamic Programming



If all nodes are close to leaves and there are many of them, then some must be identical.

$\Longrightarrow$ We can improve the running time by computing the respective solutions only once and storing them in a database.

Search Trees and Dynamic Programming

Example: Vertex Cover and the 2^k algorithm

- Each node of the search tree is an induced subgraph.
- After a reduction to problem kernel, the size of a graph is bounded by $2k'$ from above if we are looking for a solution of size k'.
- There are at most $O\left(\binom{2k}{2k'}\right)$ induced subgraphs of size at most $2k'$.
- The running time is $O^*\left(2^{k-k'} + 2^{k'}\binom{2k}{2k'}\right)$ if we store solutions of size $2k'$ in the database.
- If we choose the value of k' optimally, the running time becomes 1.886^k.