Parameterized Algorithms WS 2021
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Date: December 20th, 2021

Exercise Sheet with solutions 08

The association of elves of Santa Claus is organizing a christmas party. Every elf has some other elves that he or she is friends with, but maybe they are not friends with everyone. The organizing comitee for the christmas party has k elves, and it is very important that every elf has at least one friend in the organizing comitee if they don't belong to the comitee themselves, this ensures that their christmas cookie preferences will be taken into account. Finding a good comitee is hard, even W[2]-hard, but this year Santa Claus had an idea. The elves will be divided into green elves and red elves, to fabricate toys for good and bad children respectively. Thus, as the elves must work together in the factory, they don't want to meet up with their colleagues after work, and they only maintain friendships with elves working on the other group. Santa belives that the choice of comitee will be made easier with this friendship division. What do you think?

Task T25

The Dominating Set problem is W[2]-complete in general but in many well-known graph classes it is fixed-parameter tractable. For instance, it has a linear kernel on the class of planar graphs (and, in fact, on graphs of bounded genus, on H-minor-free graphs etc.). A colleague claims that the problem is FPT on bipartite graphs. Would you agree with your colleague? Justify your answer.

Solution

The Dominating Set problem remains W[2]-complete on bipartite graphs: there is an easy reduction from the same problem in general graphs. Given an instance (G, k) of Dominating Set in general graphs, create a bipartite graph $\tilde{G}$ as follows: create two copies of the vertex set of G and call them V_1 and V_2; let z and z' be two special vertices that are not elements of $V_1 \cup V_2$. The vertex set of $\tilde{G} = (V_1 \cup \{z'\}) \dot{\cup} (V_2 \cup \{z\})$ and the edge set of $\tilde{G}$ consists of the following edges:

1. for each $\{u, v\} \in E(G)$, add edges $\{u_1, v_2\}$ and $\{v_1, u_2\}$ in $\tilde{G}$, where u_i, v_i are the copies of u, v in V_i;
2. add edges $\{z, u_1\}$ for all $u_1 \in V_1$;
3. add the edge $\{z, z'\}$.

This completes the edge set construction of $\tilde{G}$.
We claim that G has a dominating set of size k if and only if $\tilde{G}$ has a dominating set of size $k+1$. For the forward direction, let $S \subseteq V(G)$ be a dominating set of G of size at most k. Let S_1 be the copies of these vertices in V_1. The set $S_1 \cup \{z\}$ dominates all vertices of $\tilde{G}$: vertex z dominates z' and all vertices in V_1; and S_1 dominates all of V_2.
Conversely, if S' is a dominating set of size at most $k+1$ in $\tilde{G}$, then we may assume without loss of generality that $z \in S'$. For if $z \notin S'$, it must be that $z' \in S'$ and, in this case, we can replace z by z'. Since z dominates all of V_1, we may assume that our dominating set does not contain any vertices from V_2 (because these can only dominate vertices in V_1). Thus the remaining k vertices of S' are from V_1. The originals of these vertices in G then dominate all of $V(G)$.

Task T26

You are given an $n \times n$ matrix M and an integer parameter k. The goal is to select k non-zero entries S such that every other non-zero entry is either in the same row or same column as some element in S. Is this problem in FPT or W[1]-hard? Justify your answer.

Solution

We reduce to a problem kernel.

Step 1. Delete empty rows and columns and they do not play any role in dominating others.

Step 2. If for a column c, there are at least $k+2$ copies c, keep only $k+1$ of them. Similarly, if for a row r, there are at least $k+2$ copies of r, keep only $k+1$ of them. Since we must select only k non-zero entries in the dominating set, only k columns can actually contribute to the solution and the entries of the remaining columns have to be dominated by non-zero entries in rows. Therefore it suffices to keep only $k+1$ of the columns.

Step 3. If there exist a column that is repeated $k+1$ times and has at least $k+1$ non-zero entries, then the given instance is a NO-instance.

If there is a solution, then the k dominating entries can be rearranged in the upper-left corner of the matrix. The bottom-right corner of the matrix is a block of zeros. Now there can be at most $2^k - 1$ distinct rows, each repeated $k+1$ times and the same for columns. Hence the total size of the matrix is at most $4^k(k+1)^2$.

Task T27

Prove that the Strong Exponential Time Hypothesis implies the Exponential Time Hypothesis.

Solution

copy paste proof here

Task H17 (5 credits)

The INDUCED MATCHING problem is to decide whether a given graph G has an induced matching of size at least k, where k is the parameter that is supplied as part of the input. While the MAXIMUM MATCHING problem is polynomial-time solvable, the INDUCED MATCHING problem is NP-complete in general.

Is INDUCED MATCHING W[1]-hard or FPT on regular graphs? Prove it.

Solution

Reduce from the k-INDEPENDENT SET in regular graphs. Given an instance (G, k) of k-INDEPENDENT SET where G is regular, construct a graph $\tilde{G}$ as follows: Take two copies of G, say G_1 and G_2, and for each vertex $v \in V(G)$, connect its copy $v_1 \in V(G_1)$ to the closed neighborhood $N[v_2]$ of its copy $v_2 \in V(G_2)$; and, vice versa: connect v_2 to all vertices in $N[v_1]$. Show that G has a k-independent set if and only if $\tilde{G}$ has a k-induced matching.

Task H18 (10 credits)

Show that assuming ETH implies that we cannot solve Dominating Set in time $f(k)n^{o(k)}$ for any computable function f (where k is the size of the dominating set and n the size of the graph).

Solution

Reduce from 3-sat or 3-coloring, careful with the parameter still being linear even if the running time is double exponential TODO (the book talks about this)

Hint: The christmas log (Tió), in Catalunya, can help you with this problem. Just feed him clementines, and during christmas eve you can make it sh*t by singing the following song:

Caga Tió,
petit minyó
caga neules i torrons
i bones solucions
si no vols cagar et donaré un cop de bastó,
tió tió.

it will surely “bring” you some solutions for the problem.