Parameterized Algorithms WS 2021
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Date: January 31st, 2022

Exercise Sheet with solutions 11

Task T33

Given a binary matrix of size $m \times n$, the *k-rows* problem consists in finding k rows such that their conjunction is 0, i.e., a row with n zeros.

If the k-rows problem is FPT on matrices of size $\frac{n}{2} \times n$, is the k-rows problem FPT on square matrices of (size $n \times n$)? Could you use an FPT algorithm for k-rows on $\frac{n}{2} \times n$ matrices to solve the problem on matrices of size $n \times n$?

Solution

Yes, one can build a bisecting family with size only as a function of k. An (n, k) bisecting family, given natural numbers n and k consists of a family functions $f_i\colon [n] \to \{0, 1\}$, with $|f_i^{-1}(0)| = \lceil \frac{n}{2} \rceil$ for every i, and such that for every subset $S \subseteq [n]$ with $|S| = k$, there is a function f_i with $S \subseteq f^{-1}(0)$.

If one has a bisecting family of size $g(k)$ only depending on the parameter k built in FPT time with respect to k and n, one can use this bisecting family to select $\frac{n}{2}$ rows, $g(k)$ times. One can apply then the FPT algorithm solving the k-rows problem on $\frac{n}{2} \times n$ matrices for each selection. The running time is FPT as long as the time to construct the family is FPT and the size of the family only depends on k.

Finally, we can build such a bisecting family of size 4^k as follows. Divide $[n]$ into $2k$ segments. The bisecting family consists of all possible ways to assign k segments to 0 and the other k segments to 1. This has size $\binom{2k}{k}$, which is $O^*(4^k)$. If we consider any subset S of size k, it will be distributed among the $2k$ segments in at most k of them, but for every selection of k segments there is a function mapping all of them to 0 by construction.

Task T34

A graph H is a *d-cluster graph* H if H has d connected components and every connected component is of H is a clique.

For a set of *adjacencies* $A \subseteq \binom{V(G)}{2}$, we denote with $G \oplus A$ the graph $(V(G), E(G) \triangle A)$. Show that the d-Clustering is in FPT with a subexponential funktion in k:

- Input: A graph $G = (V, E)$
- Parameter: k
- Question: Can you add or remove edges at most k edges in G such that the resulting graph is a d-cluster graph? That is, find an A of size at most k such that $G \oplus A$ is a d-cluster graph.

Lemma 1. *If the vertices of a simple graph G with k edges are colored independently and uniformly at random with $\lceil \sqrt{8k} \rceil$ colors, then the probability that $E(G)$ is properly colored is at least $2^{-\sqrt{k/2}}$.*

Solution

We use a randomized coloring approach with $q := \lceil \sqrt{8k} \rceil$ many colors on the vertices of G.

We observe the following: Let G be a graph and $\chi \colon V(G) \to [q]$ be a coloring function. Furthermore, let V_i denote the vertices of G that are colored i. If there exists a solution A in G that is properly colored by χ, then for every V_i, $G[Vi]$ is an ℓ-cluster graph for some $\ell \leq d$.

Each subgraph $G[V_i]$ is a induced subgraph of $G \oplus A$ as they do not contain colorful edges. As each induced subgraph of a cluster graph is a cluster graph, $G[V_i]$ is a cluster graph with at most d components.

With this observation, we describe an algorithm that finds A on such a coloring χ.

Every component of $G[V_i]$ is a clique which is contained completely in some component of $G \oplus A$. For each component we guess where it lays in $G \oplus A$.