Parameterized Algorithms WS 2021
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Exercise Sheet with solutions 10

Task T30

The MSO type of a structure S with a finite domain is the set of all MSO formulas ϕ with $S \models \phi$. Let us say that the q-type are the formulas in the type that have at most q variables. For simplicity we always assume that formulas are in prenex normal form.

a) Is the q-type of a structure finite or can it be infinite?

b) If it is infinite, are there only finitely many equivalence classes with regard to logical equivalence betweens formulas?

c) How could representatives of these equivalence classes look like?

Solution

a) The q-type of a structure can be infinite. If the q-type contains a formula ϕ we can construct an infinte sequence of equivalent formulas which all have at most q variables by conjuncting with true statements.

b) and c) There are at most finitely many equivalence classes. We take a formula ϕ with q variables. Then we rename these variables into $x_1, \ldots, x_q$. The quantifier free subformula is now converted into an equivalent short formula in CNF. The result is an equivalent formula whose size depends only on q.

Task T31

For a graph $G = (V, E)$ with a t-protrusion X we look at the structure $S = (V, E, X, y_1, \ldots, y_r)$ where $y_1, \ldots, y_r$ is the border of X.
Show that the following problem is in FPT for some function f:

- Input: $S = (V, E, X, y_1, \ldots, y_r)$, $t, q \in \mathbf{N}$
- Parameter: t, q
- Question: If X' is another t-protrusion with the same border as X we define $S' = (V', E', X', y_1, \ldots, y_r)$. Is there a smaller X' with $S \models \phi$ iff $S' \models \phi$ for all MSO-formulas ϕ with at most q variables?

Solution

We enumerate all t-protrusions by graph size until one of them has the correct q-type. Computing the q-type is FPT by exercise T30. As the enumeration is sorted by size and as it stops at latest when the input protrusion is enumerated, this method is correct and complete. Regarding the running time, note that for the enumeration we can "forget" about the input protrusion. Hence the number of steps of the enumeration can only depend on t and q, say it is bounded by $f(t, q)$ for some function f. Thus the problem is FPT. Please note this function is not computable.

Task T32

Prove that VERTEX COVER has finite integer index and k-PATH does not.

Solution

We give only a high-level proof idea.

It is clear that given two t-protrusions G_1 and G_2 with the same border, that $(G_1 \oplus H, k) \in$ VERTEX COVER $\iff$ $(G_2 \oplus H, k) \in$ VERTEX COVER for all H if and only if their "table entries" are identical. This follows from the fact that the boundary is a separator in the glued graphs and that Vertex Cover only talks about distance-1 properties. Hence, we want to bound the offset to bound the number of equivalence classes. Consider any "table entry" for Vertex Cover for some vertex v on the border. Now let us change this table entry such that v is included in the vertex cover. Then the size of a minimal vertex cover which fits to the new entry is at most one larger than the of the old one. As there are t vertices on the border, the range of entries is at most t. Hence, there are at most t^t many tables and hence, only finitely many equivalence classes.

For k-Path this is not possible: Let G_i be two disconnected paths of length i each where one endpoint each is in the boundary. Let H_0 be the independent set of size 2 and H_1 be the connected graph of size 2, both with both vertices in the boundary. Consider $G_3 \oplus H_0$ and $G_3 \oplus H_1$. The first graph contains a path of length at most 3, the other one of length 7. For $G_2 \oplus H_0$ and $G_2 \oplus H_1$ it is 2 and 5 respectively. Note that the differences cannot be explained by a simple offset and hence, the k-Path problem has not f.i.i.

Task H21 (10pts)

Let t be a constant. Design an efficient algorithm that solves the following problem in polynomial time:

- Input: A graph G and a number k
- Output: A t-protrusion in G of size at least k or the answer that no such protrusion exists.

The degree of a polynomial that upper bounds the running time may depend on t.

Solution

We enumerate all subsets of vertices of size at most t. We remove the boundary from the graph and check if one of the components has size at least k and treewidth at most t. The treewidth of a graph can be recognized in FPT-time with parameter t. The subsets can be enumerated in $O(n^t)$.

Task H22 (10pts)

Find a graph class that excludes some H as a topological minor, but contains *every* graph H as a minor (i.e., contains a graph that has H as a minor).

Solution

Let K_5 be the clique with 5 vertices. If a graph G contains K_5 as a topological minor then G contains vertices with degree at least 4. We need to construct a family of graphs of degree at most 3 which contains every graph H as a minor. One such family is the family of all 3-regular graphs.