**RWTH Aachen**
**Theoretical Computer Science**
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Exercise 10
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# Exercise for Analysis of Algorithms

## Exercise 37

Matthias lives in an old German building *(Altbau)* and has one of these typical guest bathrooms which are narrow but very long ($2 \times n$). He wants to improve its outdated look by paving the floor with new tiles. Luckily, he found a bunch of tiles which fell from the back of a truck. Since the tiles are in a specific color pattern he does not know how he can pave this strip so that it looks best. The tiles are (identical) $1 \times 2$ tiles he can use and rotate by 90 degrees: ▯. His approach is to try every tiling by laying it out and to give it a beauty-value. One such tiling needs $O(n)$ time. He thinks that, if the exponential growth of the total time needed is less than $4.5^n$, he can find the most beautiful bathroom tiling without loosing his mind. Can he find it or is he doomed?

**Solution:**

Let $A$ be the number of configurations for the strip. We use the symbolic method to find a generating function.

A = | + ▯ A + ⊟ A

The generating function is thus

$$A(z) = 1 + zA(z) + z^2A(z)$$
$$A(z) = \frac{1}{1 - z - z^2}$$

To estimate the exponential growth we have to find the root of this function.

$$z^2 + z - 1 = 0$$

Note that we have to find the smallest positive root.

$$z_{1,2} = -\frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + 1}$$
$$z_1 = -\frac{1}{2} + \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2} - \frac{1}{2} \approx 0.61803$$

The exponential growth is thus given by $(\frac{1}{z_1})^n = 1.61804^n$ and find his favorite tiling in a “fast” manner.

**Exercise 38**
Matthias found somewhere else a new tile shape, namely an $1 \times 1$ tile: $\square$. This increases the options and the time needed for him to try out all possibilites. Can he still find the best looking option in time if he also considers the new tile?

**Solution:**
Let us now count the area and not the length of a tile, that is $\Box$ for example has now weight 2 and no longer 1. We have $A$ for the configurations for a strip as before

$$A = | + {}^{\square}B + {}_{\square}B + \big[\,\big]A + \boxminus A$$

and $B$ for configurations that are missing a $\square$ tile either in the top left or bottom left (the two cases are symmetric).

$$B = {}^{\square}A + {}^{\sqsubset\!\sqsupset}B$$

Which gives rise to the following generating functions

$$A(z) = 1 + 2zB(z) + z^2A(z) + z^4A(z)$$
$$B(z) = zA(z) + z^2B(z)$$

We rewrite the lower one as

$$B(z) = \frac{zA(z)}{1 - z^2}$$

And substitute it in the first one

$$A(z) = 1 + \frac{2z^2A(z)}{1 - z^2} + z^2A(z) + z^4A(z)$$
$$A(z) = \frac{1}{1 - z^2 - z^4 - \frac{2z^2}{1-z^2}}$$
$$A(z) = \frac{1 - z^2}{(1 - z^2)(1 - z^2 - z^4) - 2z^2}$$

We solve

$$(1 - z^2)(1 - z^2 - z^4) - 2z^2 = 0$$

and get as dominant singularity $\alpha \approx 0.504$. Since the area is $2n$ we get an exponential growth of $(\frac{1}{\alpha})^{2n} = (\frac{1}{0.504})^{2n} = 3.9366^n$ for the number of tilings for an $n \times 2$ bathroom, which luckily is still enough to find it in time.

**Exercise 39**
Find the exponential growth of the following functions:

a) $2^n n^3$

b) $\left(\frac{2}{3}\right)^{2n} + 5$

c) $[z^n]\frac{1}{\sqrt{1-5z}}$

d) $[z^n]\frac{z^2-1}{(z-1)(z-5)}$

e) $[z^n]\frac{1}{e-e^{3/2-z^2}}$

**Solution:**

a) $2^n$

b) $1$

c) $5^n$

d) $(1/5)^n$

e) $\sqrt{2}^n$

We only show two solutions as a) is kind of easy and c) - e) are very similar.

b) Two show that $\left(\frac{2}{3}\right)^{2n} + 5 \asymp 1$. We check the first condition of the alternative definition: $\left(\frac{2}{3}\right)^{2n} + 5 > (1-\varepsilon)^n$ as $(1-\varepsilon)^n < 5$ for every $n > 0$. The second condition $\left(\frac{2}{3}\right)^{2n} + 5 < (1+\varepsilon)^n$ holds once $(1+\varepsilon)^n$ reaches 6 for every subsequent $n$. This happens as the function is strictly monotone increasing.

d) We use the theorem linking the exponential growth to the dominant singularity of its generating function. Every singularity of $\frac{z^2-1}{(z-1)(z-5)}$ has to fulfill $(z-1)(z-5) = 0$ which holds for $z = 1$ and $z = 5$. However, $z = 1$ is *not* a singularity as the function can be continued at this undefined point (the limit at this points exits). But 5 is a singularity. Because it is the only one, it is the dominant singularity. Hence $\frac{z^2-1}{(z-1)(z-5)} \asymp (1/5)^n$.

Note that the generating function in e) has many singularities on the complex plane but the dominant singularity (of power series with non-negative coefficients) is always real.

**Exercise 40**

Find very large and very small functions with exponential growth 1, 0 and $\infty$.

**Solution:**

exponential growth 1:

- $(1/1000)^{n/\log(n)}$
- $(1/2)^{\sqrt{n}}$
- $(1/n)^3$
- $1$
- $n^3$
- $2^{\sqrt{n}}$
- $1000^{n/\log(n)}$

exponential growth 0:

- $0$
- $1/n!$
- $(1/2)^{n^2}$
- $(0.999)^{n\log(n)}$

exponential growth $\infty$:

- $(1.001)^{n\log(n)}$
- $2^{2^n}$

**Exercise 41**

Sort the following generating functions *within one minute* by their exponential growth!

1. $A(z) = \frac{1}{\sqrt{1-z/2}}$
2. $B(z) = \frac{1}{1-e^{z-1/3}}$
3. $C(z) = \frac{1+z}{1-z}$

**Solution:**

We only need to sort them by the absolute value of the dominant singularities. $A_n \asymp 1/2^n$, $B_n \asymp 3^n$, $C_n \asymp 1$. Therefore $A_n \leq C_n \leq B_n$.

**Exercise 42**

In this exercise we will look at 2-3-trees. They are rooted, ordered trees. Each internal node has either two or three children. As usual, the size of a 2-3-tree will be the number of its internal nodes.

1. How can you define 2-3-trees recursively?
2. Enumerate all 2-3-trees of size two. How many are there? How many trees exist of sizes zero and one?
3. Find a generating function $Q(z)$ for the number $q_n$ of 2-3-trees with size $n$.
4. What is the dominant singularity of $Q(z)$ and what is the exponential growth of $q_n$? Use a computer algebra system. Do not give up when you see horrifying formulas.

**Solution:**

We get immeadiately the equation $Q(z) = zQ(z)^2 + zQ(z)^3 + 1$. If we solve it with the help of a computer algebra system, we get a very messy result, but we can spot the expression $\sqrt{z^2 + 11z - 1}$, which defines one the singularities. It seems that it is probably the dominant one. If that is true, then $\alpha = (5^{3/2} - 11)/2$ is the singularity for which we are looking. The exponential growth is then $\alpha^{-n} \approx 11.09016994374933^n$.

A closer look shows that this is indeed the dominant singularity.