RWTH Aachen **WS 2025**
Theoretical Computer Science Exercise 9
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Exercise for Analysis of Algorithms

Exercise 31

Compute:

$$\text{(a) } [z^n]\frac{1}{1+2z} \qquad \text{(b) } [z^n]\frac{z+1}{z-1} \qquad \text{(c) } [z^n]\Big(\frac{z+1}{z-1}\Big)^2 \qquad \text{(d) } [z^n]\frac{1}{\sqrt[3]{5+z}}$$

Solution:

a)
$\sum_{n\geq 0} \alpha^n z^n$, yields $\frac{1}{1-\alpha z}$. So $[z^n]\frac{1}{1+2z} = (-2)^n$.

b)
We use that

$$\frac{n+1}{n-1} = 1 - 2\frac{1}{1-n}$$

and get $A(z) - 2B(z)$ with

$$[z^n]A(z) = (n = 0)$$

and

$$[z^n]B(z) = 1$$

to obtain $[z^n]\frac{z+1}{z-1} = (n = 0) - 2$.

c)
The convolution rule yields

$$A(z)^2 = \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} a_k a_{n-k}\Big) z^n$$

We use our solution for a_n from b) to obtain For $n \neq 0$

$$\sum_{k=0}^{n} a_k a_{n-k} = \Big(\sum_{k=1}^{n-1} a_k a_{n-k}\Big) + a_0 \cdot a_n + a_n \cdot a_0 = 4(n-1) + 4 = 4n$$

and for $n = 0$

$$\sum_{k=0}^{n} a_k a_{n-k} = 1$$

Which yields $[z^n]\Big(\frac{z+1}{z-1}\Big)^2 = 4n + (n = 0)$

d) We start with

$$\frac{1}{\sqrt[3]{5+z}} = \frac{1}{\sqrt[3]{5}}\frac{1}{\sqrt[3]{1+z/5}}$$

We use that

$$[z^n](1+z)^r = \binom{r}{n}$$

Finally we use scaling with 1/5 to obtain

$$[z^n]\frac{1}{\sqrt[3]{5+z}} = \frac{1}{\sqrt[3]{5}}\binom{-\frac{1}{3}}{n}5^{-n}$$

Exercise 32

How many subsets of $\{1,\dots,2000\}$ have a sum divisible by 5?

With generating functions at hand, you are able to solve this exercise. As this question seems to have a bit different nature than the questions we usually look at during our lecture, you have to think a bit outside of the box.

These could be guiding questions for you: For which sequence (g_n) do you want to get a generating function? How can you get the answer for the exercise from the generating function? For this questions, maybe consider the (much easier) case where you want to know the number of subsets with a sum divisible by 2.

Solution:

This exercise is inspired by a video by the Youtber 3Blue1Brown:

Olympiad level counting: How many subsets of 1,...,2000 have a sum divisible by 5?
`https://www.youtube.com/watch?v=bOXCLR3Wric`

We can only recommend you to watch the video. You can skip to 6:51 where the generating function is introduced.

You can be very creative in solving this exercise, so I will show you multiple solutions:

Using generating functions We consider the generating function $G(z) = \sum_{n=0}^{\infty} g_n z^n$ where g_n denotes the number of subsets of [2000] whose sum is n. Note that only finitely many coefficients are non-zero.

Let us consider the how $G(z)$ looks like for some special values of z. The value of $G(1)$ is the number of all subsets, so @^{2000}, as $G(1) = \sum_n g_n$. $G(-1)$ is the alternating sum of the subsets, i.e., $\sum_n (-1)^n g_n$ which is 0. Interestingly, $\frac{1}{2}(G(1)+g(-1))$ is the number of all subsets with an *even* sum due to cancellation.

We can extend this cancellation trick to get the number of subsets with a sum divisible by 5, i.e., $g_0, g_5, g_{10}, \dots$. We want to find ζ, such that

$$[z^n]\frac{1}{5}(G(\zeta^0)+G(\zeta^1)+G(zeta^2)+G(\zeta^3)+G(\zeta^4)) = \begin{cases} g_n \cdot 1 & \text{if } 5 \mid n \\ 0 & \text{if } 5 \nmid n \end{cases}.$$

This means, we want ζ For the case above, ζ was -1. Here, we need the 5th roots of unity, namely the solutions to $z^5 - 1 = 0$.

Using recurrences and summation factor First of all, we consider $\{0,1,2,3,4\}$ as we only care for the numbers modulo 5. With f_m^d we denote the number of combinations $\{0,1,2,3,4\}$ after m of up to d module 5. We get this recurrence relation for $d=0$: $f_0^0 = 1$ (empty set) and

$$f_m^0 = 8f_{m-1}^0 + 6f_{m-1}^1 + 6f_{m-1}^2 + 6f_{m-1}^3 + 6f_{m-1}^4$$

Note that $f_m^1 + f_m^2 + f_m^3 + f_m^4 = 2^{5m} - f_m$. Hence, $f_m^0 = 8f_{m-1}^0 + 6(2^{5(m-1)} - f_{m-1})$. We can use summation factor and get (after som calculations): $f_m =$
Then f_400 yields the result.
Or using other combinatorical tricks

Exercise 33
Find a function $f(n)$ in closed form such that

$$\prod_{k=1}^{n} k^k = f(n)(1 + O(1/n^2)).$$

Use Euler summation. It is okay if you cannot find the correct constant in the sum.

Solution:
Similar to $n!$ we turn to logarithms in order to turn the product into a sum and get immeadiately (with $f(x) = x \ln x$)

$$\ln\Big(\prod_{k=1}^{n} k^k\Big) = \sum_{k=1}^{n} k \ln k \sim \int_1^n x \ln x \, dx + \frac{n \ln n}{2} + C + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(n).$$

We need to find out what the integral and what the derivatives of f are. We get $\int x \ln x \, dx = x^2(2\ln x - 1)/4 + C$ and $f'(x) = ln(x) + 1$, $f''(x) = 1/x$, and $f^{(k)}(x) = (-1)^k (k-2)! x^{-k+1}$ for $k \geq 2$. This gives us

$$\sum_{k=1}^{n} k \ln k \sim \frac{n^2(2\ln n - 1)}{4} + \frac{n \ln n}{2} + C' + \frac{1/6}{2!} \ln(n) + \sum_{k=2}^{\infty} \frac{B_{2k}(-1)^{2k-1}(2k-3)!}{(2k)! n^{2k-2}}$$
$$= \sum_{k=1}^{n} k \ln k = \frac{n^2 \ln n}{2} - \frac{n^2}{4} + \frac{n \ln n}{2} + \frac{\ln n}{12} + C' - \frac{1}{720n^2} - \frac{1}{5040n^4} \pm \cdots$$

Now that we have an asymptotic expansion of the logarithm of the desired product we just have to apply the exponential function to it and we get:

$$\bar{\sigma} n^{n^2/2 - n^2/4 \ln n + n/2 + 1/12}(1 + O(1/n^2))$$

Unfortunately, we do not know the constant $\bar{\sigma}$ yet...

Exercise 34
If you use Euler summation on a polynomial function, can you get an *exact* solution? Prove it or find a counterexample.

Solution:
It is indeed possible to get an exact solution using Euler summation on a polynomial function. If we take a look at the Euler summation formula it states that:
If $\int_1^n |f^{(i)}(x)| \, dx$ exists for $1 \leq i \leq 2m$, then

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x) \, dx + \frac{1}{2} f(n) + C + \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(n) + R_m,$$

where $R_m = O\left(\int_1^n |f^{(2m)}(x)|\, dx\right)$ and $B_k = n![z^n]z/(e^z - 1)$ are the Bernoulli-numbers. We see that we have an error term that depends on the $2m$-th derivative of f. If we think about the derivatives of polynomials we quickly notice that if we derive a polynomial of degree t, $t+1$ times, the derivative becomes 0, making the error term disappear if we choose m in such a way that for a polynomial of degree t, $t \leq 2m$. So, we obtain an exact formula.

Moreover, the other non-exact term we have is the additive constant, which we can obtain by substituting for instance for $n = 1$ on that particular polynomial but it is even easier than that.

If we realize that both sides of the equation are polynomials we can even substitute for $n = 0$. This makes the left hand side 0 thus we realize that the constant term of the polynomial on the right hand side must be 0. This does not mean that the constant C is 0 but that when we add C to the other constants that appear when applying the formula we get 0.

Exercise 35

Approximate the following sum up to an error of $O(n^{-5})$:

$$\sum_{k=1}^{n} \frac{1}{k^2}$$

Find the constant C in Euler's summation formula by looking up $\sum_{k=1}^{\infty} \frac{1}{k^2}$. Test your result for $n = 1000$. Use your favorite computing software.

Solution:

Euler's summation formula gives us

$$\sum_{k=1}^{n} \frac{1}{k^2} = \int_1^n \frac{dx}{x^2} + \frac{1}{2n^2} + C + \frac{B_2}{2}\left(-\frac{2}{n^3}\right) + R_2.$$

Moreover, it holds that

$$R_2 = O\Big(\int_n^\infty \Big| \left(\frac{1}{x^2}\right)^{(4)} \Big|_{x=n} \Big|\, dx\Big) = O(n^{-5}),$$

and

$$\sum_{k=1}^{n} \frac{1}{k^2} = \frac{\pi^2}{6} - \frac{1}{n} + \frac{1}{2n^2} - \frac{1}{6n^3} + O(n^{-5})$$

because of this identity

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \zeta(2) = \frac{\pi^2}{6}.$$

The value for $n = 1000$ is $1.64393456668156\ldots$. We get these results

$\frac{\pi^2}{6}$	$\underline{1.64}49\ldots$
$\frac{\pi^2}{6} - \frac{1}{n}$	$\underline{1.643934}0\ldots$
$\frac{\pi^2}{6} - \frac{1}{n} + \frac{1}{2n^2}$	$\underline{1.643934566}8\ldots$
$\frac{\pi^2}{6} - \frac{1}{n} + \frac{1}{2n^2} - \frac{1}{6n^3}$	$\underline{1.64393456668156}\ldots$

Exercise 36
Approximate the following sum up to an error of $O(n^{-5})$:

$$\sum_{k=1}^{n} \frac{1}{k^{5/2}}$$

Solution:
Euler's summation formula gives us

$$\sum_{k=1}^{n} \frac{1}{k^{5/2}} = \int_{1}^{n} \frac{1}{k^{5/2}} dx + \frac{1}{2n^{5/2}} + C + \frac{B_2}{2}(-\frac{5/2}{n^{7/2}}) + R_2.$$

Moreover, it holds that

$$R_2 = O\Big(\int_{n}^{\infty} \Big| (\frac{1}{x^{5/2}})^{(4)} \Big|_{x=n} \Big| \; dx\Big) = O(n^{-5}),$$

and

$$\sum_{k=1}^{n} \frac{1}{k^2} = 1.3419 - \frac{2}{3n^{3/2}} + \frac{1}{2n^{5/2}} - \frac{5}{24n^{7/2}} + O(n^{-5})$$

because of this identity

$$\sum_{k=1}^{\infty} \frac{1}{k^{5/2}} = \zeta(2.5) = 1.3419...$$