RWTH Aachen
Theoretical Computer Science
Rossmanith–Gehnen

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Exercise 5
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Exercise for Analysis of Algorithms

Exercise 17

Solve the following recurrence relation by order reduction:

$$a_0 = 8000 \qquad a_1 = \frac{1}{2} \qquad a_{n+2} + a_{n+1} - n^2 a_n = n!$$

Solution:

$$a_{n+2} + a_{n+1} - n^2 a_n = (E - n)\underbrace{(E + n)a_n}_{b_n}$$

Hence we have to solve the recurrence equation for b_n:

$$\begin{aligned} &(E - n)b_n = n! \\ \iff\ & b_{n+1} - nb_n = n! \end{aligned}$$

We guess the solution $b_n = n!$. We insert that into the recurrence relation for a_n an get:

$$\begin{aligned} &(E + n)a_n = b_n = n! \\ \iff\ & a_{n+1} + na_n = n! \end{aligned}$$

We get the following solution for a_n

$$a_n = \frac{(n-1)!}{2}$$

for $n > 0$ and $a_0 = 8000$.
To find the solution to $b_n = (n-1)b_{n-1} + (n-1)!$ one can use summation factors. We get

$$\begin{aligned} b_n &= (n-1)! + \sum_{j=1}^{n-1} (j-1)! \cdot j \cdot (j+1) \cdots (n-1) \\ &= (n-1)! + (n-1)(n-1)! = n!. \end{aligned}$$

Solving $a_n = -(n-1)a_{n-1} + (n-1)!$ is similar.

Exercise 18

Solve the following recurrence relation:

$$a_0 = 0 \qquad a_1 = 1 \qquad a_{n+2} + a_{n+1} - n^2 a_n = n!$$

This is the same recurrence relation as in the last task, but the initial conditions are different. You can either use order reduction as in the last task, but you can also choose whatever method you like.

Solution:
We use the repertoire method as we already know the solution for the original recurrence relation. Hence, we have in our repertoire $a'_n = (n-1)!/2$ with values $a'_0 = 0$, $a'_1 = 1/2$ and $f(n) = n!$. Note that we only need to find a solution, which changes the value for a_1. It is therefore sufficient to find some solution to the corresponding homogeneous recurrence, i.e., with $f(n) = 0$.
After some time we try $b_n := (-1)^n(n-1)!/2$. Then $b_1 = 1/2$ and $f(n) = 0$:

$$\begin{aligned} b_{n+2} + b_{n+1} - n^2 b_n = (-1)^{n+2}\frac{(n+1)!}{2} + (-1)^{n+1}\frac{n!}{2} - (-1)^n n^2 \frac{(n-1)!}{2} \\ = \frac{1}{2}(-1)^n\big((n+1)n! - n! - n\cdot n!\big) = 0. \end{aligned}$$

The solution for the recurrence relation with $a_1 = 1$ is $a_n = b_n + a'_n$ because $b_1 = 1/2$ and $a'_1 = 1/2$.

Exercise 19
Solve the following recurrence relation:

$$a_n = n + 1 + \frac{1}{n}\sum_{k=0}^{n-1} a_k \text{ for } n > 0 \text{ and } a_0 = 2$$

Solution:
If we apply the repertoire method, we can quickly realize that for $a_n = 1$, $a_n - \frac{1}{n}\sum_{k=0}^{n-1} a_k = 0$, which means that 1 is a solution to the homogeneous part of the recurrence. For $a_n = n$ we obtain $a_n - \frac{1}{n}\sum_{k=0}^{n-1} a_k = (n+1)/2$. To obtain the desired recurrence we take $a_n = 2n+2$, where the first part gives us the inhomogeneous term and the second part accomodates for the initial condition.
Alternatively, one can look at the first few terms, guess the solution to be $2n + 2$ and then prove its correctness by induction.

Exercise 20
How often is the loop in the following excerpt executed if $0 < i$ holds at the beginning?

```
while i <= j
i := i+j;
j :=j+10;
```

Solution:
We denote the value of i in the nth repetition by i_n (and similar for j_n). For $i_0 > j_0$, the while-loop is never executed. Let thus $0 < i_0 \leq j_0$. We obtain the recursion

$$\begin{aligned} i_n &= i_{n-1} + j_{n-1} \\ j_n &= j_{n-1} + 10 \end{aligned}$$

which yields (by insertion)

$$\begin{aligned} j_n &= j_0 + 10n \\ i_n &= i_{n-1} + 10(n-1) + j_0 \\ &= i_0 + \sum_{k=1}^{n} (10(k-1) + j_0) \\ &= i_0 + 5n(n-1) + nj_0. \end{aligned}$$

The loop is executed as long as $i_n - j_n \leq 0$, which implies

$$5n^2 + (j_0 - 15)n + i_0 - j_0 \leq 0.$$

We know that for a polynomial of degree two holds

$$ax^2 + bx + c = 0 \iff x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

For positive n we therefore need

$$n \leq \frac{15 - j_0 + \sqrt{(j_0 - 15)^2 - 20(i_0 - j_0)}}{10} =: a(i_0, j_0)$$

holds. In this case, the loop is hence executed $\lfloor a(i_0, j_0) \rfloor + 1$ times.