RWTH Aachen | **WS 2025**
Theoretical Computer Science | Exercise 4
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Exercise for Analysis of Algorithms

Exercise 14

Given an array a of length n, an algorithm compares all pairs $(a[i], a[j])$ for all $i < j \leq n$, and then calls itself recursively on all proper prefixes of a.
How often does the algorithm compare two pairs? Use the repertoire method!

Solution:
The recurrence is $R_0 = 0$ and

$$R_n = \binom{n}{2} + \sum_{k=0}^{n-1} R_k$$

for $n \geq 1$. Testing a couple of values for R_n, we obtain the repertoire:

R_0	R_n	$f_n = R_n - \sum_{k=0}^{n-1} R_k$
1	a^n	$\frac{2-a}{1-a}a^n - \frac{1}{1-a}$
1	2^n	1
1	1	$1 - n$
0	n	$n - \binom{n}{2}$

To get $\binom{n}{2}$, we can use the last line. To get rid of the linear and constant factors, we use the third and finally the second line, and obtain

$$-\left(n - \binom{n}{2}\right) - (1 - n) + 1 = \binom{n}{2}.$$

Fortunately, this also holds for $R_0 = 0$, and the solution is

$$R_n = 2^n - n - 1.$$

Exercise 15

Use the repertoire method to find a closed form for the following recurrence:

$$\begin{aligned} a_0 &= 5 \\ a_1 &= 9 \\ a_n &= na_{n-1} + n^2 a_{n-2} - n^4 - 3n^2 + 5 \qquad \text{for } n \geq 2 \end{aligned}$$

Solution:
Let $f(n) = -n^4 - 3n^2 + 5$, i.e., $f(n) = a_n - na_{n-1} - n^2 a_{n-2}$.

a_n	$f(n)$	a_0	a_1
1	$-n^2 - n + 1$	1	1
n	$-n^3 + n^2 + 2n$	0	1
n^2	$-n^4 + 3n^3 - n^2 - n$	0	1

Let Z_i for $i = 1, 2, 3$ be the solutions of the first, second, and third line, respectively. Then $f(n) = 5Z_1 + 3Z_2 + Z_3$. For these, a_0 and a_1 are correct, and thus $a_n = 5 \cdot 1 + 3n + n^2$.

Exercise 16
Use summation factors to solve the following recurrence:

$$\begin{aligned} a_0 &= 0 \\ a_n &= \frac{a_{n-1}}{n} + \frac{1}{(n-1)!} \qquad \text{for } n \geq 1 \end{aligned}$$

Solution:
Plugging $y_n = 1/(n-1)!$ and $x_n = 1/n$ into the formula known from the lecture yields:

$$\begin{aligned} a_n &= \frac{1}{(n-1)!} + \sum_{j=1}^{n-1} \frac{1}{(j-1)!} \frac{1}{j+1} \cdots \frac{1}{n} \\ &= \frac{1}{(n-1)!} + \sum_{j=1}^{n-1} \frac{j}{n!} \\ &= \frac{1}{(n-1)!} + \frac{1}{n!} \frac{n(n-1)}{2} \end{aligned}$$