Analysis of Algorithms WS 2022
Prof. Dr. P. Rossmanith
M. Gehnen, H. Lotze, D. Mock

Date: 16 February 2023

Old Exam (2020) with solutions 0

This is an old exam from 2020.

Task K1 (10 Points)

Order the following four power series by their asymptotic growths. Justify your answer.

a) $[z^n]e^{z+z^2}$

b) $[z^n]e^{z+z^2/2}$

c) $[z^n]\sqrt{1-z-z^2}$

d) $[z^n]1/\sqrt{1-z-z^2}$

Solution

The series defined by a) and b) have no singularities, thus c) and d) grow faster than a) and b). Moreover, a) grows faster than b) because e^{z^2} grows faster than $e^{z^2/2}$.

Both c) and d) have singularities when $1-z-z^2=0$, which happens when $z=-1/2\pm\sqrt{5}/2$, and the singularity at $z=(\sqrt{5}-1)/2$ is dominant, which means that both have an exponential growth of $(2/(\sqrt{5}-1))^n$.

Because the dominant singularities of both c) and d) are algebraic, we can be more precise by using Theorem 10 and be see that c) grows as $O(n^{-3/2}(2/(\sqrt{5}-1))^n)$ and d) as $O(n^{-1/2}(2/(\sqrt{5}-1))^n)$ with lower order terms. So d) grows faster than c), and we get the final ordering: $d>c>a>b$

Task K2 (10 Points)

Solve the following recurrence relation:

$$a_n = n+1+\frac{1}{n}\sum_{k=0}^{n-1} a_k \text{ for } n>0 \text{ and } a_0=2$$

Solution

If we apply the repertoire method, we can quickly realize that for $a_n=1$, $a_n-\frac{1}{n}\sum_{k=0}^{n-1}a_k=0$, which means that 1 is a solution to the homogeneous part of the recurrence. For $a_n=n$ we obtain $a_n-\frac{1}{n}\sum_{k=0}^{n-1}a_k=(n+1)/2$. To obtain the desired recurrence we take $a_n=2n+2$, where the first part gives us the inhomogeneous term and the second part accomodates for the initial condition.

Alternatively, one can look at the first few terms, guess the solution to be $2n+2$ and then prove its correctness by induction.

Task K3 $(1 + 7 + 2$ Points)
Consider the following context-free grammar G:

$$S \to aSbS \mid cSdS \mid \epsilon$$

a) Write down all words up to length four of $L(G)$.

b) Find out whether the number of words of length up to n grows asymptotically faster or slower than 3^n. Justify your answer.

c) The generating function has two dominant singularities on the real axis. Explain why this is normally not the case but happens here.

Solution
a) The words of length up to 4 are:

$$\epsilon, ab, cd, aabb, abab, acdb, abcd, cabd, cdab, ccdd, cdcd$$

b) The words of G can be expressed by this symbolic expression:

$$S = \{a\} \times S \times \{b\} \times S \dot{\cup} \{c\} \times S \times \{d\} \times S \dot{\cup} \{\epsilon\}$$

(Note that the grammar is unambiguous and hence the unions are disjoint.)

From that we get directly an expression for the generating function $S(z)$:

$$S(z) = z^2 S(z)^2 + z^2 S(z) + 1$$

We solve for $S(z)$

$$\iff S(z)^2 - \frac{1}{2z^2} S(z) + \frac{1}{2z^2} = 0$$

$$\iff S(z) = \frac{1}{4z^2} \pm \sqrt{\left(\frac{1}{4z^2}\right)^2 - \frac{1}{2z^2}}$$

The generating function has (dominant) singularities at $z = \pm\frac{1}{\sqrt{8}} = \pm\frac{1}{2\sqrt{2}}$. Hence the exponential growth of S_n is $(2\sqrt{2})^n \leq 2.83^n$ and it grows asymptotically slower than 3^n.
c) The number of words for uneven is 0. Hence there are two singularities to make infinitely many coefficients equal zero.

Task K4 (10 Points)
Consider the problem *Triangle Deletion:*
Input: A graph G and budget $k \in \mathbf{N}$.
Output: Yes iff there is a set $W \subseteq V(G)$ and a set $F \subseteq E(G)$ such that $2|W| + |F| \leq k$ and $G - W - F$ is triangle-free.
We propose the following branching algorithm $\mathcal{A}(G, k)$ for this problem.

1. If $k < 0$, return NO.
2. If G is triangle-free, return YES.
3. Otherwise, find a triangle $\{v_0, v_1, v_2\}$ in G.
4. Call $\mathcal{A}(G - v_i, k - 2)$ for each $0 \leq i \leq 2$.
5. Call $\mathcal{A}(G - e, k - 1)$ for each edge $e \in \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_1\}\}$.
6. If any of the recursive calls returns YES, return YES. Otherwise NO.

Analyze the number of recursive calls in the worst-case for a given budget k. The exponential growth of the number of recursive calls is precise enough.

Solution

In the worst case, the algorithm makes three calls with $k-1$ and three with $k-2$. Hence we geht this recurrence relation:

$$c_k = 3c_{k-1} + 3c_{k-2}$$

The characteristic polynomial for that is

$$\chi(z) = z^2 - 3z - 3$$

The roots of it are $3/2 \pm \frac{\sqrt{2}}{2}$.

Hence the dominant singularity is at $3/2 - \frac{\sqrt{2}}{2} = -0.792$ (other one is at 3.79) and the exponential growth $\frac{1}{3/2-\frac{\sqrt{2}}{2}}^n = (-1.27)^n$.