Analysis of Algorithms WS 2022
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Exercise Sheet with solutions 12

Tutorial Exercise T12.1

Let

$$U(z) := \frac{1 - z - \sqrt{(1-3z)(1+z)}}{2z}.$$

Prove that $[z^n]U(z) = 3^n n^{O(1)}$ without doing any computations. Then find out what the constant in the monomial is, i.e., for what c is $[z^n]U(z) = \Theta(n^c 3^n)$.

Solution

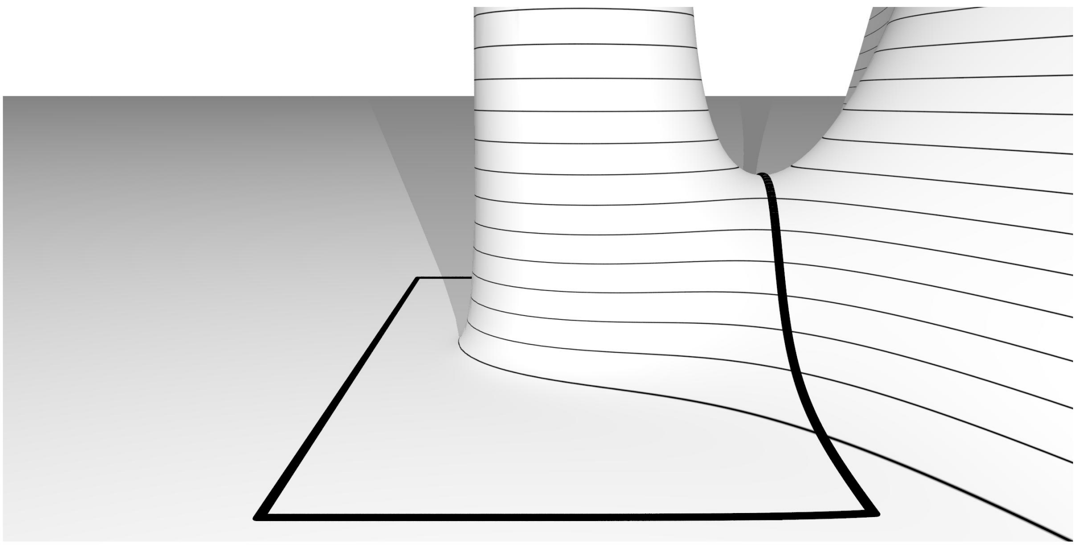
The dominant singularity 1/3 is an algebraic singularity of order $c = 1/2$. Therefore $[z^n]U(z) = \Theta(n^{-c-1}3^n) = \Theta(3^n/n^{3/2})$.

Tutorial Exercise T12.2

In the lecture we used the saddle point method to approximate $[z^n]e^z$. In order to do it, we chose a circle as our integrating path.
Approximate now $[z^n]e^z$ using the same method but choosing a rectangular integrating path.
In order to simplify the calculation, you can use a degenerated rectangle.

Solution

We



choose a rectangle that goes through the points $\pm iD$, $-S$ and R. At the beginning have no need to fix the value of D and S, but we do have to fix $R = n + 1$, in order for the integrating path to go through the saddle point. The integral we need to calculate goes through the four edges of the rectangle, and we separate it in those four parts. Let us name them $I_1, \ldots, I_4$.

The first thing we observe is that if we take the limit value $D \to \infty$ then both integrals

$$I_2 = \int_{R+iD}^{-S+iD} \frac{e^z}{z^{n+1}} dz \text{ and } I_4 = \int_{-S-iD}^{R-iD} \frac{e^z}{z^{n+1}} dz \ ,$$

go to zero.
It is also clear that

$$I_3 = \int_{-S-iD}^{-S+iD} \frac{e^z}{z^{n+1}} dz = O(e^{-S}) \ ,$$

so if we set $S = n$, then we can ignore the term I_3 too.
So the only interesting integral is the remaining one

$$I_1 = \int_{R-iD}^{R+iD} \frac{e^z}{z^{n+1}} dz,$$

which we divide again into two parts

$$A = \int_{-\delta}^{\delta} \frac{e^{R+it}}{(R+it)^{n+1}} i\, dt$$

and

$$B = \int_{-\infty}^{\delta} \frac{e^{R+it}}{(R+it)^{n+1}} i\, dt + \int_{\delta}^{\infty} \frac{e^{R+it}}{(R+it)^{n+1}} i\, dt.$$

We first aproximate A:

$$\begin{aligned} iA = \int_{-\delta}^{\delta} \frac{e^{R+it}}{(R+it)^{n+1}}\, dt &= \frac{e^R}{R^{n+1}} \int_{-\delta}^{\delta} e^{it-(n+1)\ln(1+it/R)} dt = \\ &= \frac{e^R}{R^{n+1}} \int_{-\delta}^{\delta} e^{t^2/2(n+1)} \big(1 + O(t^3/n^2)\big)\, dt = \frac{e^R}{R^{n+1}} \big(1 + O(\delta^3/n^2)\big) \int_{-\delta}^{\delta} e^{t^2/2(n+1)}\, dt \end{aligned}$$

We choose δ such that $\delta^3/n^2 = o(1)$, and obtain

$$iA = \frac{e^R}{R^{n+1}} \sqrt{2(n+1)\pi} \ .$$

Through and simple subsitution and appending the tails (the value of B), we can take this to be the integral from $-\infty$ to ∞.
We obtain then finally

$$1/n! = [z^n]e^z = \frac{1}{2\pi i} \oint \frac{e^z}{z^{n+1}} dz \sim \frac{1}{2\pi} A = \frac{1}{2\pi} \frac{e^R}{R^{n+1}} \sqrt{2(n+1)\pi}$$

Homework Exercise H12.1
We continue exercise T12.1 where

$$U(z) = \frac{1 - z - \sqrt{(1-3z)(1+z)}}{2z}.$$

and we found the constant c with $[z^n]U(z) = \Theta(n^c 3^n)$.
Now also find the multiplicative constant in the Θ-notation, i.e., find a simple function $f(n)$ such that $[z^n]U(z) \sim f(n)$.

Solution
We estimate $U(z)$ in the vicinity of the dominant singularity:

$$U(z) \sim \frac{1 - \frac{1}{3} - \sqrt{(1-3z)(1+\frac{1}{3})}}{2 \cdot \frac{1}{3}} = 1 - \sqrt{3}\sqrt{1-3z} \text{ for } z \to \frac{1}{3}$$

The theorem about algebraic singularities says that

$$[z^n]U(z) \sim -\frac{\sqrt{3}\, n^{-3/2}\, 3^n}{\Gamma(-1/2)} = \frac{\sqrt{3}\, n^{-3/2}\, 3^n}{2\sqrt{\pi}}.$$

The following exercise is from a former exam:

Homework Exercise H12.2
Consider the following context-free grammar G:

$$S \to aSbS \mid cSdS \mid \epsilon$$

a) Write down all words up to length four of $L(G)$.

b) Find out whether the number of words of length up to n grows asymptotically faster or slower than 3^n. Justify your answer.

c) The generating function has two dominant singularities on the real axis. Explain why this is normally not the case but happens here.

Solution
a) The words of length up to 4 are:

$$\epsilon, ab, cd, aabb, abab, acdb, abcd, cabd, cdab, ccdd, cdcd$$

b) The words of G can be expressed by this symbolic expression:

$$S = \{a\} \times S \times \{b\} \times S \dot\cup \{c\} \times S \times \{d\} \times S \dot\cup \{\epsilon\}$$

(Note that the grammar is unambiguous and hence the unions are disjoint.)

From that we get directly an expression for the generating function $S(z)$:

$$S(z) = z^2 S(z)^2 + z^2 S(z) + 1$$

We solve for $S(z)$

$$\iff S(z)^2 - \frac{1}{2z^2} S(z) + \frac{1}{2z^2} = 0$$
$$\iff S(z) = \frac{1}{4z^2} \pm \sqrt{\left(\frac{1}{4z^2}\right)^2 - \frac{1}{2z^2}}$$

The generating function has (dominant) singularities at $z = \pm\frac{1}{\sqrt{8}} = \pm\frac{1}{2\sqrt{2}}$. Hence the exponential growth of S_n is $(2\sqrt{2})^n \leq 2.83^n$ and it grows asymptotically slower than 3^n.
c) The number of words for uneven is 0. Hence there are two singularities to make infinitely many coefficients equal zero.