Analysis of Algorithms WS 2022
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Date: December 19, 2022

Exercise Sheet with solutions 08

Due date: next tutorial session, preferably in groups

Tutorial Exercise T8.1

Here is a *classical* problem: n gentlemen attend a Christmas party and check their hats. As it can happen with *gentlemen* at Christmas parties, they have a little too much drink and the checker returns the hats at random. What is the probability that no gentlemen receives his own hat? How does the probability depend on the number of gentlemen?

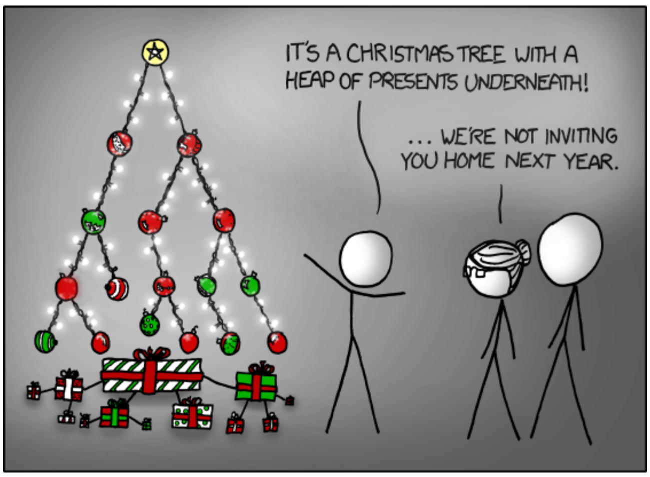
Solution

Let the n gentlemen be labeled $1, 2, \ldots, n$. A permutation of $\{1, \ldots, n\}$ in which element i is not placed at position i, for any i, is called a *derangement*. For example, for $n = 3$, 312 is a derangement but 321 is not as 2 is in the second place.

Let D_n denote the number of derangements of n elements. Clearly $D_1 = 0$. $D_2 = 1$ as 21 is the only derangement. We will define $D_0 = 1$. It is convenient to say that there is one permutation of the empty set and that it does not map anything to itself.

Consider the general case with $n + 1$ elements. Element 1 has to be at some position k, where $2 \leq k \leq n + 1$. Now there are two possibilities. Either element k is at position 1, in which case there are D_{n-1} derangements possible. Otherwise, some other element is at position 1. This second situation may also be viewed as follows: We keep element 1 fixed at the first position; derange elements $2, \ldots, n + 1$ in D_n ways; finally, exchange the elements at the first and kth positions to obtain a derangement of the elements $1, \ldots, n + 1$. The recurrence for D_{n+1} may now be written as:

$$D_{n+1} = n(D_n + D_{n-1}). \tag{1}$$



xkcd 835: *Not only is that terrible in general, but you just KNOW Billy's going to open the root present first, and then everyone will have to wait while the heap is rebuilt.*

Using the above recurrence, we can write $D_{n+1} - (n+1)D_n$ as:

$$\begin{aligned} D_{n+1} - (n+1)D_n &= nD_{n-1} - D_n \\ &= -\left(D_n - nD_{n-1}\right) \\ &= (-1)^2\left(D_{n-1} - (n-1)D_{n-2}\right) \\ &= (-1)^3\left(D_{n-2} - (n-2)D_{n-3}\right) \\ &\;\vdots \\ &= (-1)^{n-1}(D_2 - 2D_1). \end{aligned}$$

Put differently, the recurrence (1) may be expressed as:

$$D_{n+1} = (n+1)D_n + (-1)^{n+1} \quad \text{where} \quad n \geq 2. \tag{2}$$

Define $D(z) = \sum_{n=0}^{\infty} D_n \frac{z^n}{n!}$. Multiply both sides by $z^{n+1}/(n+1)!$ and sum over n, obtaining:

$$\sum_{n=0}^{\infty} D_{n+1}\frac{z^{n+1}}{(n+1)!} = \sum_{n=0}^{\infty}(n+1)D_n\frac{z^{n+1}}{(n+1)!} + \sum_{n=0}^{\infty}(-1)^{n+1}\frac{z^{n+1}}{(n+1)!}$$

The left-hand-side is $D(z) - D_0$. The first term on the right-hand-side is $zD(z)$ and the second term is $e^{-z} - 1$. Thus the above equation may be written as:

$$D(z) = \frac{e^{-z}}{1-z} = \sum_{n=0}^{\infty}(-1)^n\frac{z^n}{n!}\sum_{n=0}^{\infty} z^n,$$

from which we may write down D_n as:

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n\frac{1}{n!}\right).$$

Thus the probability that no gentleman recieves his own hat is $D_n/n!$ which approaches $e^{-1} = 0.3678\ldots$ This is independent of n.

Tutorial Exercise T8.2

Find a bivariate generating function and a closed-form expression for the number of bitstrings of length n that contain exactly m ones and do not contain the substring 11.

Solution

Let $b_{n,m}$ be the number of bitstrings that do not contain 11, have length n, and contain m ones. Such bitstrings can be described by the recursive definition $F = 0F + 10F + \varepsilon + 1$.

We get the following generating function:

$$
\begin{aligned}
F(u,z) = & \sum_{n,m\geq 0} b_{n,m} u^m z^n \\
= & zF(z) + uz^2F(z) + 1 + uz \\
= & \frac{1+uz}{1-z-uz^2} \\
= & (1+uz)\sum_{k\geq 0} z^k(1+uz)^k \\
= & (1+uz)\sum_{k\geq 0} z^k \sum_{i=0}^{k} \binom{k}{i} u^i z^i \\
= & (1+uz)\sum_{k\geq 0}\sum_{i=0}^{k} \binom{k}{i} u^i z^{i+k} \\
= & \sum_{k\geq 0}\sum_{i=0}^{k} \binom{k}{i} u^i z^{i+k} + \sum_{k\geq 0}\sum_{i=0}^{k} \binom{k}{i} u^{i+1} z^{i+k+1}
\end{aligned}
$$

We are interested in the coefficients corresponding to $z^n u^m$. Thus we need to consider the summand with $i = m \wedge i + k = n$ (front) and $i + 1 = m \wedge i + k + 1 = n$ (back). This gives the closed formula for the number of valid bitstrings:

$$\binom{n-m}{m} + \binom{n-m}{m-1} = \binom{n-m+1}{m}$$

Homework Exercise H8.1

Bots became quite creative these days. During the Christmas season, they will spam users with Christmas greetings of the following form:

$$P \to \quad 🎅 P ❄️ \quad | \quad 🎄 P 🕯️ \quad | \quad ❤️ \quad | \quad ❤️ P$$

How many unique messages of length n can you get at most? Particularly interesting values are $n = 4096$, as these are the maximal lengths of text messages on WhatsApp and Telegram. Use the symbolic method. Emojis are the terminal symbols and capital letters are variables.

Solution

The language of this grammar can be described by the following recursive definition.

$$P = (\{😀\} \times P \times \{😍\}) \cup (\{🤡\} \times P \times \{🙀\}) \cup \{💩\} \cup (\{💩\} \times P)$$

We need to define the weight of the atomic elements.

$$|😀| = |😍| = |🤡| = |🙀| = |💩| = 1$$

Let $T(z)$ be the generating function for the number of words of length n generated by the grammar. The symbolic method yields

$$T(z) = 2z^2T(z) + z + zT(z).$$

We transform this into

$$T(z) = \frac{z}{1-z-2z^2}.$$

Notice that $1 - z - 2z^2 = (z+1)(1-2z)$. We want to find a partial fraction decomposition of the form

$$\frac{z}{1-z-2z^2} = \frac{A}{z+1} + \frac{B}{1-2z}.$$

By setting $z = 0$ we get $0 = A + B$ and by setting $z = 1$ we get $-1/2 = A/2 - B$. Together they yield $A = -1/3$ and $B = 1/3$. This means

$$T(z) = -\frac{1}{3}\frac{1}{z+1} + \frac{1}{3}\frac{1}{1-2z}$$

and therefore $[z^n]T(z) = \frac{1}{3}(2^n - (-1)^n)$. There are $\frac{1}{3}(2^n - (-1)^n)$ words of length n.

Homework Exercise H8.2

We will again look at the question from H3.2:

How many subsets of $\{1, \dots, 2000\}$ have a sum divisible by 5?

With generating functions at hand, you are able to solve this exercise. As this question seems to have a bit different nature than the questions we usually look at during our lecture, you have to think a bit outside of the box.

These could be guiding questions for you: For which sequence (g_n) do you want to get a generating function? How can you get the answer for the exercise from the generating function? For this questions, maybe consider the (much easier) case where you want to know the number of subsets with a sum divisible by 2.

Solution

This exercise is inspired by a video by the Youtber 3Blue1Brown:

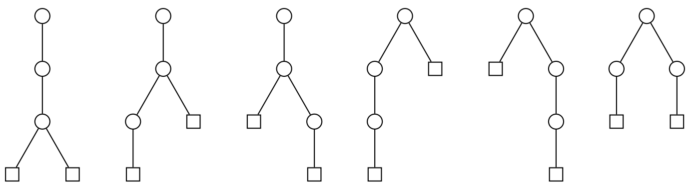
Olympiad level counting: How many subsets of 1,. . . ,2000 have a sum divisible by 5?
`https://www.youtube.com/watch?v=bOXCLR3Wric`

I can only recommend you to watch the video. You can skip to 6:51 where the generating function is introduced.

Homework Exercise H8.3

Find a bivariate generating function for the number of rooted, oriented trees with exactly n internal and m external vertices $T_{n,m}$. For what values of n, m do we have $T_{n,m} = T_{m,n}$?

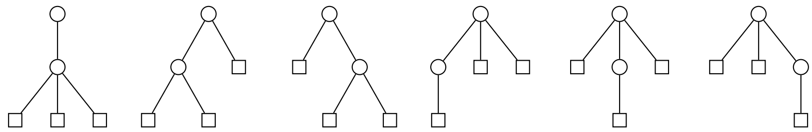
Example: $b_{3,2} = 6$ and these are the six trees with 3 internal and 2 external nodes:



As a warmup exercise try to find and draw all trees with 2 internal and 3 external nodes.

Hint: Do not do all the computations by hand. Seek the help of a computer algebra system. `maxima` or `WolframAlpha` can solve quadratic equations and can find the coefficients of a generating function via Taylor expansion.

Solution
We can draw first all trees with 2 internal and 3 external nodes as a warmup.



And we see that $b_{2,3} = b_{3,2} = 6$.
We consider a recursive definition of trees. A tree is either an external node or an internal node with at least one subtree. This yields

$$T = \cup \bigcirc \times T \times T^*$$

und and the generating function

$$T(u, z) = u + z\frac{T(u, z)}{1 - T(u, z)},$$

where u is the number of external and z is the number of internal nodes. We need to solve this equation for $T(u, z)$. We delegate this task to `maxima`: The call `solve(T=u+z*T/(1-T),T)` yields

$$-\frac{\sqrt{z^2 + (-2 * u - 2) * z + u^2 - 2 * u + 1} + z - u - 1}{2}$$

and

$$+\frac{\sqrt{z^2 + (-2 * u - 2) * z + u^2 - 2 * u + 1} - z + u + 1}{2}.$$

We know that for $u = z = 0$ the solution needs to be 0. Thus only the first solution is correct. We get the coefficients by doing a Taylor expansion of $T(u, z)$. We enter

```
T:-((sqrt(z^2+(-2*u-2)*z+u^2-2*u+1)+z-u-1)/2);
taylor(T,[z,u],0,5);
```

into `maxima`. We read the coefficients:

$n + m$	Term
1	u
2	uz
3	$uz^2 + u^2z$
4	$uz^3 + 3u^2z^2 + u^3z$
5	$uz^4 + 6u^2z^3 + 6u^3z^2 + u^4z$

We also want to find out for which values holds $T(u, z) = T(z, u)$. We do so by calculating $T(u, z) - T(z, u)$. We type into `maxima`:

```
TT(u,z):=-((sqrt(z^2+(-2*u-2)*z+u^2-2*u+1)+z-u-1)/2);
ev(TT(u,z)-TT(z,u),expand);
```

The answer is $u - z$. Therefore, the generating function of the difference is $u - z$. The all coefficients of this function are zero except for the case $u = 1, z = 0$ or $u = 0, z = 1$. This means $T_{u,z} = T_{z,u}$ for all other values. This makes sense as there is exactly one tree with a single external node and zero trees with a single internal node.