Analysis of Algorithms, WS 2020
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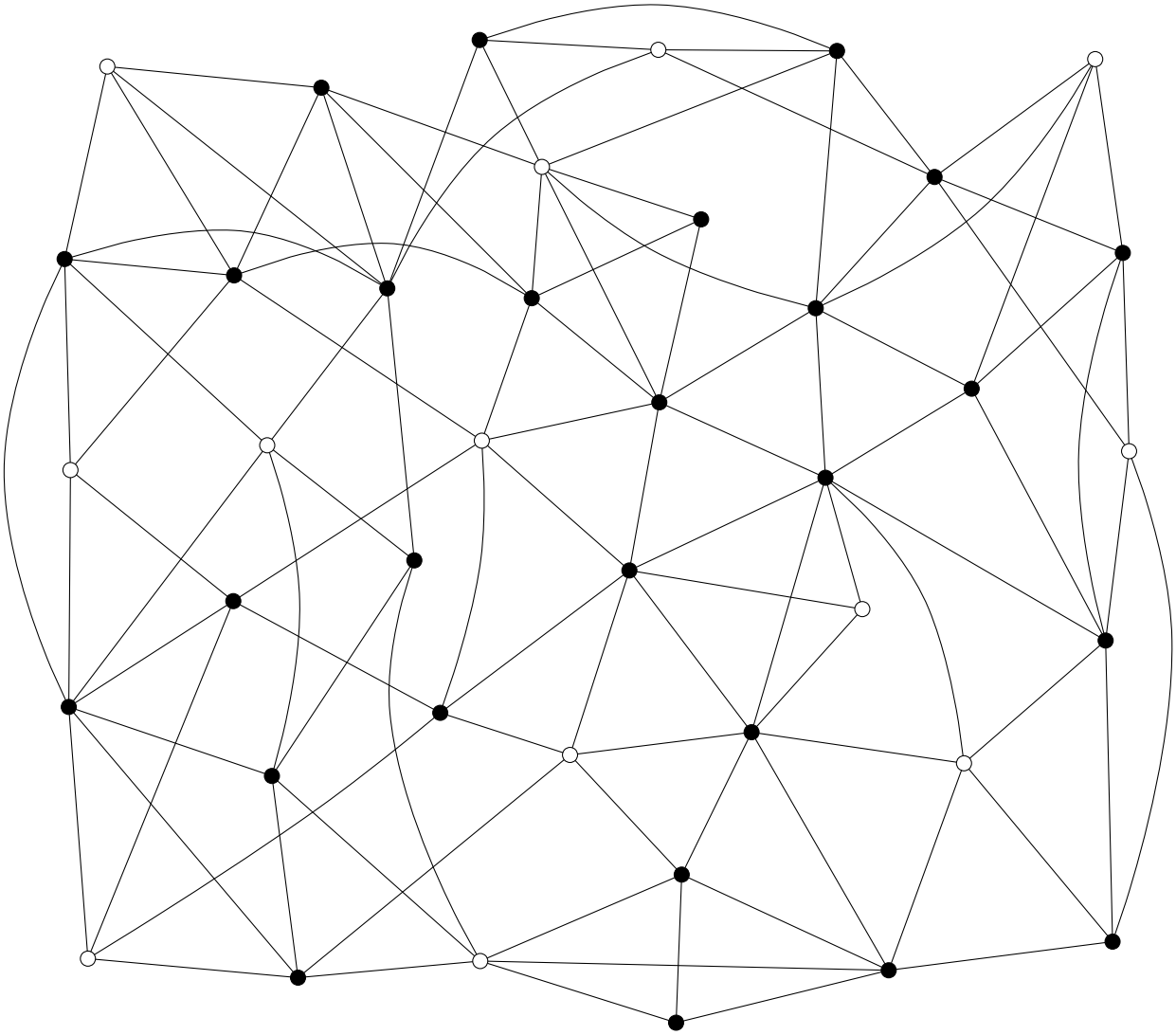
Date: 11 November 2020

Exercise Sheet with solutions 03

Problem T7

A *vertex cover* of an undirected graph $G = (V, E)$ is a subset $C \subseteq V$ of its vertices such that at least one endpoint of every edge is in C, i.e., for every $(v_i, v_j) \in E$, either $v_i \in C$ or $v_j \in C$. Informally speaking, "C covers all edges." It is an NP-complete problem to find out whether a graph has a vertex cover of a given size. The following example shows a graph of order 40. The black vertices comprise a minimum size vertex cover.



Now, given a graph $G = (V, E)$ we define its size as the number of vertices $|V|$ of the graph. So, let us consider a graph of size n and let k be targeted vertex cover size. Find an algorithm for Vertex Cover that runs in time $1.5^k n^{O(1)}$. Hints: If a graph has maximum degree two, i.e., without any vertex of degree three or more, this problem can be solved in polynomial time. On the other hand, if a vertex v is not in the vertex cover, all of its neighbors have to be there.

Solution

Let us denote by a_k the running time of our algorithm if the targeted size of the vertex cover

is k. Given a graph $G = (V, E)$ of size n, first we check that there is a vertex of degree higher than two. As the hint states, if a given graph has degree two, one can solve it in polynomial time. Otherwise, let us take a vertex v of degree 3 or more, and let $N(v) \subseteq V$ be its neighborhood. We do the following branching procedure.

If we take the vertex v into the vertex cover, we only need to find a vertex cover of size $k-1$ on the graph without v or its neighbors, i.e., $G \setminus \{v, N(v)\}$. This will have a running time of a_{k-1}.

Otherwise, if v is not the vertex cover, then, as the hint says, $N(v)$ needs to be in the vertex cover, and $N(v)$ contains at least three vertices. So, one will ned to find a vertex cover of size $k-3$ on the same graph $G \setminus \{v, N(v)\}$ as before. This will have a running time of at most a_{k-3}.

This gives us the recurrence relation $a_k = a_{k-1} + a_{k-3}$ for the dependence in k. Additionally one can take into account the dependence in n, which is linear in each iteration, so we know that in total it will be a polynomial term multiplying the term with the dependency in k. So, in order to know the running time of this algorithm, we only need to solve the recurrence relation we just found by using the usual methods.

The recurrence relation $a_k = a_{k-1} + a_{k-3}$ is linear and has constant coefficients, its characteristic polynomial is $\chi(z) = z^3 - z^2 - 1$. The roots of this polynomial can be found using the closed formula for cubic polynomials or a solver and they are

$$\alpha_1 = \frac{1}{3}\left(1 + \sqrt[3]{\frac{1}{2}(29 - 3\sqrt{93})} + \sqrt[3]{\frac{1}{2}(29 + 3\sqrt{93})}\right),$$
$$\alpha_2 = \frac{1}{3} - \frac{1}{6}(1 - i\sqrt{3})\sqrt[3]{\frac{1}{2}(29 - 3\sqrt{93})} - \frac{1}{6}(1 + i\sqrt{3})\sqrt[3]{\frac{1}{2}(29 + 3\sqrt{93})},$$
$$\alpha_3 = \frac{1}{3} - \frac{1}{6}(1 + i\sqrt{3})\sqrt[3]{\frac{1}{2}(29 - 3\sqrt{93})} - \frac{1}{6}(1 - i\sqrt{3})\sqrt[3]{\frac{1}{2}(29 + 3\sqrt{93})}.$$

If we want an exact formula we can take the initial terms a_0, a_1, and a_2 and find the coefficients of the generic solution $a_k = \mu_1(\alpha_1)^k + \mu_2(\alpha_2)^k + \mu_3(\alpha_3)^k$. This is complicated, however, because the roots of this polynomial are not simple. Moreover, we might not need an exact solution; if we look at the approximate values we see that $\alpha_1 \approx 1.46$, $\alpha_2 \approx -0.2 - 0.8i$, and $\alpha_3 \approx -0.2 + 0.8i$. Because α_2 and α_3 are much smaller than α_1 in terms of absolute value, the first root dominates and we can say that $a_k = O(1.5^k)$, which means that the given algorithm for vertex cover achieves the targeted running time.

Problem T8

Solve the recurrence relation

$$a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3}$$

with the starting conditions $a_0 = 1$, $a_1 = 3$, $a_2 = 8$.

Solution

The characteristic polynomial is $\chi(z) = z^3 - 5z^2 + 8z - 4 = (z-2)^2(z-1)$. It has the double root 2 and the single root 1. Hence, the solution looks like $\alpha 2^n + \beta n 2^n + \gamma$. If $n = 0$, then $\alpha + \gamma = 1$, if $n = 1$ then $2\alpha + 2\beta + \gamma = 3$, and if $n = 2$ then $4\alpha + 8\beta + \gamma = 8$. Solving this set of equations yields $\alpha = 1/2$, $\beta = 1$, $\gamma = 0$ and the solution is therefore $a_n = (1 + n/2)2^n$.

Problem H6 (10 credits)

Solve the following recurrence: Let $a_0 = 1$, $a_1 = 1$, $a_2 = 4$ and

$$a_n = 2a_{n-1} - a_{n-2} + 2a_{n-3}, \text{ for } n \geq 3.$$

Solution

The characteristic polynomial is $z^3 - 2z^2 + z - 2$. This can be factorized as $(z^2 + 1)(z - 2)$, which gives the set of roots as $\{\pm i, 2\}$. The general solution is thus:

$$a_n = \alpha \cdot 2^n + \beta \cdot (-i)^n + \gamma \cdot i^n.$$

Substituting the values for $n = 0, 1, 2$, we obtain the following system of linear equations:

$$\alpha + \beta + \gamma = 1 \tag{1}$$
$$2\alpha - i\beta + i\gamma = 1 \tag{2}$$
$$4\alpha - \beta - \gamma = 4 \tag{3}$$

Solving this system yields $\alpha = 1$, $\beta = -i/2$ and $\gamma = i/2$. Note that β and γ are complex conjugates. The solution is thus:

$$a_n = 2^n - \frac{i}{2} \cdot (-i)^n + \frac{i}{2} i^n \tag{4}$$
$$= 2^n + (1 + (-1)^{n+1}) \cdot \frac{i^{n+1}}{2}. \tag{5}$$

This solution can be written in a much nicer form using Euler's formula:

$$e^{i\theta} = \cos\theta + i \sin\theta.$$

We may write $i^{n+1}/2$ as follows:

$$\frac{i^{n+1}}{2} = \frac{1}{2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{n+1}$$
$$= \frac{e^{\frac{i\pi}{2}\cdot(n+1)}}{2}.$$

Similarly, $(-1)^{n+1} i^{n+1}/2$ may be written as $\frac{1}{2} \cdot e^{-\frac{i\pi}{2}\cdot(n+1)}$. Now note that

$$\frac{e^{\frac{i\pi}{2}\cdot(n+1)} + e^{-\frac{i\pi}{2}\cdot(n+1)}}{2} = \cos\left(\frac{\pi(n+1)}{2}\right).$$

Thus $a_n = 2^n + \cos\frac{\pi(n+1)}{2}$.

Problem H7 (10 credits)

Our task is to generate a word of length n over the alphabet $\{0, 1\}$, which contains neither two consecutive zeros nor three consecutive ones.

Daniel proposes the following algorithm: The algorithm generates a word of length n uniformly at random. If the word fulfills the property, it is returned. Otherwise, the algorithm tries again until it finds one.

What is the expected number of rounds the algorithm needs? Consider this in particular for 32bit words.

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0110110110110101  0110110101011010  0110101101011011  0110101010110110  0101101101011011
0110110110110110  0110110101011011  0110101101010101  0110101010101101  0101101101010101
0110110110101101  0110110101010101  0110101101010110  0110101010101010  0101101101010110
0110110110101010  0110110101010110  0110101011011010  0110101010101011  0101101011011010
0110110110101011  0110101101101101  0110101011011011  0101101101101101  0101101011011011
0110110101101101  0110101101101010  0110101011010101  0101101101101010  0101101011010101
0110110101101010  0110101101101011  0110101011010110  0101101101101011  0101101011010110
0110110101101011  0110101101011010  0110101010110101  0101101101011010  0101101010110101
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0101101010110110  0101010101010101  1101011010110110  1011011010101010  1010110101011011
0101101010101101  0101010101010110  1101011010101101  1011011010101011  1010110101010101
0101101010101010  1101101101101101  1101011010101010  1011010110110101  1010110101010110
0101101010101011  1101101101101010  1101011010101011  1011010110110110  1010101101101101
0101011011011010  1101101101101011  1101010110110101  1011010110101101  1010101101101010
0101011011011011  1101101101011010  1101010110110110  1011010110101010  1010101101101011
0101011011010101  1101101101011011  1101010110101101  1011010110101011  1010101101011010
0101011011010110  1101101101010101  1101010110101010  1011010101101101  1010101101011011
0101011010110101  1101101101010110  1101010110101011  1011010101101010  1010101101010101
0101011010110110  1101101011011010  1101010101101101  1011010101101011  1010101101010110
0101011010101101  1101101011011011  1101010101101010  1011010101011010  1010101011011010
0101011010101010  1101101011010101  1101010101101011  1011010101011011  1010101011011011
0101011010101011  1101101011010110  1101010101011010  1011010101010101  1010101011010101
0101010110110101  1101101010110101  1101010101011011  1011010101010110  1010101011010110
0101010110110110  1101101010110110  1101010101010101  1010110110110101  1010101010110101
0101010110101101  1101101010101101  1101010101010110  1010110110110110  1010101010110110
0101010110101010  1101101010101010  1011011011011010  1010110110101101  1010101010101101
0101010110101011  1101101010101011  1011011011011011  1010110110101010  1010101010101010
0101010101101101  1101011011011010  1011011011010101  1010110110101011  1010101010101011
0101010101101010  1101011011011011  1011011011010110  1010110101101101  1010101010110101
0101010101101011  1101011011010101  1011011010110101  1010110101101010  1010101010110110
0101010101011010  1101011011010110  1011011010110110  1010110101101011  1010101010101010
0101010101011011  1101011010110101  1011011010101101  1010110101011010  1010101010101011
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Solution

We say a word is correct if it contains neither two consecutive zeros nor three ones. We count the number of correct words of length n to derive the probability that a random word is correct. With that it is easy to compute the expected number of rounds the algorithm needs.

Let a_n be the number of correct words of length n and b_n the number of words among them that start with a 1. For b_n we get the recurrence $b_n = b_{n-2} + b_{n-3}$ for $n \geq 3$ because such a word can start either with 10 or 110, followed again by a word that starts with a 1. In general a word starts with either with 01 or with 1 giving us the recurrence $a_n = b_{n-1} + b_n$ for $n \geq 2$. As a_n is a linear combination of b_n and b_{n-1} every solution for b_n is also a solution for a_n. Hence, $a_n = a_{n-2} + a_{n-3}$ for $n \geq 4$.

Let us solve this recurrence. We compute the basic cases by hand: $a_0 = 1$, $a_1 = 2$, $a_2 = 3, a_3 = 4, a_4 = 5, a_5 = 7, a_6 = 9$.

The characteristic polynomial is $z^3 - z - 1$ with the roots

$$z_1 = \frac{\frac{\sqrt{3}i}{2} + \frac{-1}{2}}{3\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}} + \frac{1}{2}\right)^{\frac{1}{3}}} + \left(\frac{\sqrt{23}}{23^{\frac{3}{2}}} + \frac{1}{2}\right)^{\frac{1}{3}} \left(\frac{-1}{2} - \frac{\sqrt{3}i}{2}\right),$$

$$z_2 = \frac{\frac{-1}{2} - \frac{\sqrt{3}i}{2}}{3\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}} + \frac{1}{2}\right)^{\frac{1}{3}}} + \left(\frac{\sqrt{23}}{23^{\frac{3}{2}}} + \frac{1}{2}\right)^{\frac{1}{3}} \left(\frac{\sqrt{3}i}{2} + \frac{-1}{2}\right),$$

$$z_3 = \left(\frac{\sqrt{23}}{23^{\frac{3}{2}}} + \frac{1}{2}\right)^{\frac{1}{3}} + \frac{1}{3\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}} + \frac{1}{2}\right)^{\frac{1}{3}}}$$

Finding the right linear combination of basic solutions gives us the closed form in Figure 1, which is almost useless in this complicated form.

The roots of the characteristic polynomial read as follows as approximate decimal fractions:

$$z_1 = -0.5622795120623012i - 0.6623589786223729$$
$$z_2 = +0.5622795120623012i - 0.6623589786223729$$
$$z_3 = 1.324717957244746$$

$$a_n = -\frac{\left(\left(53^{\frac{21}{2}}\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}}+\frac{1}{2}\right)^{6}-73^{\frac{21}{2}}\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{17}{3}}-203^{\frac{19}{2}}\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{16}{3}}+853^{\frac{17}{2}}\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{14}{3}}+143^{\frac{19}{2}}\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{13}{3}}-293^{\frac{17}{2}}\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}}+\frac{1}{2}\right)^{4}-563^{\frac{15}{2}}\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{11}{3}}-293^{\frac{15}{2}}\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{10}{3}}+143^{\frac{15}{2}}\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}}\cdots\right.\right.\right.}{\cdots}$$

$$+\frac{\left(\left(53^{\frac{21}{2}}\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}}+\frac{1}{2}\right)^{6}-73^{\frac{21}{2}}\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{17}{3}}-203^{\frac{19}{2}}\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{16}{3}}+853^{\frac{17}{2}}\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{14}{3}}+143^{\frac{19}{2}}\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{13}{3}}-293^{\frac{17}{2}}\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}}+\frac{1}{2}\right)^{4}-563^{\frac{15}{2}}\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{11}{3}}-293^{\frac{15}{2}}\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{10}{3}}+143^{\frac{15}{2}}\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}}+\frac{1}{2}\right)^{3}+\cdots\right.\right.}{3\cdots}$$

$$+\frac{\left(\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{1}{3}}+\frac{1}{3\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{1}{3}}}\right)^{n}\left(1215\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{8}{3}}+1701\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{7}{3}}+1782\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}}+\frac{1}{2}\right)^{2}+567\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{5}{3}}+135\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{4}{3}}\right)}{729\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}}+\frac{1}{2}\right)^{4}+1215\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{10}{3}}+891\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{8}{3}}+378\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}}+\frac{1}{2}\right)^{2}+99\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{4}{3}}+15\left(\frac{\sqrt{23}}{23^{\frac{3}{2}}}+\frac{1}{2}\right)^{\frac{2}{3}}+1}$$

Fig. 1: This is the closed formula for a_n.

Only the real root is bigger than one. The absolute values of the other two are smaller than 0.87. Hence, $a_n = \alpha z_3^n + O(0.87^n)$ with $\alpha = 1.678735602594163\ldots$ and $z_3 = 1.324717957244746\ldots$. We can approximate a_{32} as

$$a_{32} = 1.678735602594163 \cdot 1.324717957244746^{32} \approx 13581.003,$$

while the exact value is $a_{32} = 13581$.
As there are 2^32 many binary words of length 32, the probability to find one is only $13581/2^32 \approx .00000316$. As the number of required rounds is geometrically distributed we need about 316248 rounds on average to find the first good word. Unfortunately this is not a very efficient method for long words.