Analysis of Algorithms, WS 2020
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Date: 3 February 2021

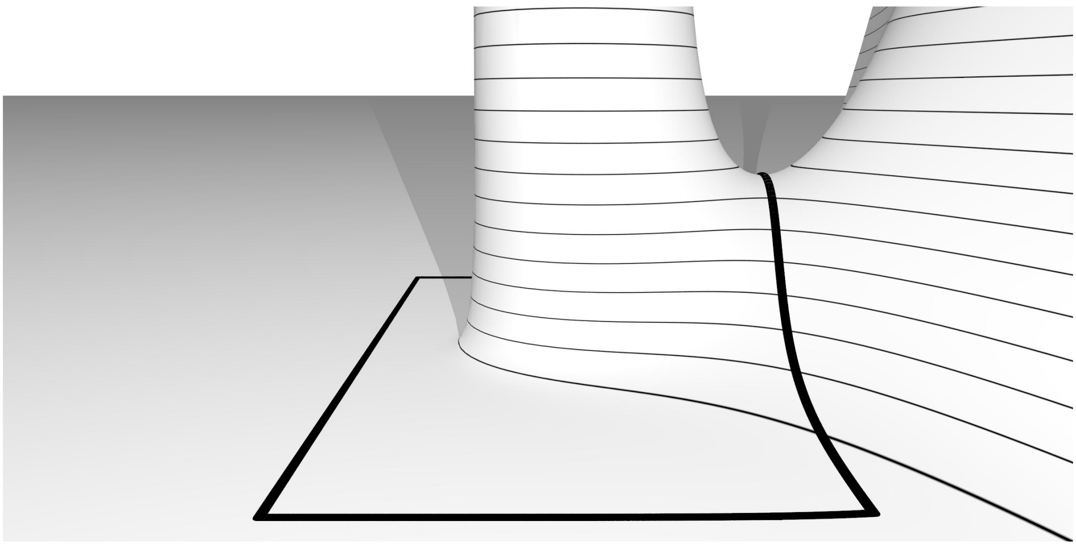
Exercise Sheet with solutions 12

Problem T28

In the lecture we used the saddle point method to approximate $[z^n]e^z$. In order to do it, we chose a circle as our integrating path.
Approximate now $[z^n]e^z$ using the same method but choosing a rectangular integrating path. In order to simplify the calculation, you can use a degenerated rectangle.

Solution

We



choose a rectangle that goes through the points $\pm iD$, $-S$ and R. At the beginning have no need to fix the value of D and S, but we do have to fix $R = n+1$, in order for the integrating path to go through the saddle point. The integral we need to calculate goes through the four edges of the rectangle, and we separate it in those four parts. Let us name them $I_1, \ldots, I_4$.
The first thing we observe is that if we take the limit value $D \to \infty$ then both integrals

$$I_2 = \int_{R+iD}^{-S+iD} \frac{e^z}{z^{n+1}} dz \text{ and } I_4 = \int_{-S-iD}^{R-iD} \frac{e^z}{z^{n+1}} dz ,$$

go to zero.
It is also clear that

$$I_3 = \int_{-S-iD}^{-S+iD} \frac{e^z}{z^{n+1}} dz = O(e^{-S}) ,$$

so if we set $S = n$, then we can ignore the term I_3 too.
So the only interesting integral is the remaining one

$$I_1 = \int_{R-iD}^{R+iD} \frac{e^z}{z^{n+1}} dz,$$

which we divide again into two parts

$$A = \int_{-\delta}^{\delta} \frac{e^{R+it}}{(R+it)^{n+1}} i\, dt$$

and

$$B = \int_{-\infty}^{\delta} \frac{e^{R+it}}{(R+it)^{n+1}} i\, dt + \int_{\delta}^{\infty} \frac{e^{R+it}}{(R+it)^{n+1}} i\, dt.$$

We first aproximate A:

$$iA = \int_{-\delta}^{\delta} \frac{e^{R+it}}{(R+it)^{n+1}}\, dt = \frac{e^R}{R^{n+1}} \int_{-\delta}^{\delta} e^{it-(n+1)\ln(1+it/R)} dt =$$
$$= \frac{e^R}{R^{n+1}} \int_{-\delta}^{\delta} e^{t^2/2(n+1)} \left(1 + O(t^3/n^2)\right) dt = \frac{e^R}{R^{n+1}} \left(1 + O(\delta^3/n^2)\right) \int_{-\delta}^{\delta} e^{t^2/2(n+1)}\, dt$$

We choose δ such that $\delta^3/n^2 = o(1)$, and obtain

$$iA = \frac{e^R}{R^{n+1}} \sqrt{2(n+1)\pi}\ .$$

Through and simple subsitution and appending the tails (the value of B), we can take this to be the integral from $-\infty$ to ∞.
We obtain then finally

$$1/n! = [z^n]e^z = \frac{1}{2\pi i} \oint \frac{e^z}{z^{n+1}} dz \sim \frac{1}{2\pi} A = \frac{1}{2\pi} \frac{e^R}{R^{n+1}} \sqrt{2(n+1)\pi}$$

Problem H28 (10 credits)
In this exercise we consider the following (regular) CFG G:

$$S \to abA \mid bS \mid a$$
$$A \to bA \mid aS$$

1. Find a generating function for number of words s_n in $L(G)$ that have length n.
2. What is the dominant singularity and what kind of singularity is it?
3. What is the exponential growth of s_n?
4. How precisely can you estimate s_n with just the knowledge of the dominating singularity and its nature?
5. Find a closed formula for s_n with an additive error of at most $O(0.8^n)$.

Solution

1. Since the grammar is unique, the symbolic method gives us

$$S(z) = z^2 A(z) + zS(z) + z,$$

$$A(z) = zA(z) + zS(z).$$

We solve the latter for $A(z)$ and obtain $A(z) = zS(z)/(1-z)$. Now we can insert it into the former. This yields

$$S(z) = z^3 S(z)/(1-z) + zS(z) + z.$$

We then solve for $S(z)$ and get the generating function

$$S(z) = \frac{z}{1 - z - z^3/(1-z)} = \frac{z(1-z)}{(1-z)^2 - z^3}.$$

2. The singularities are the roots of of the denominator. We ask Maxima `solve((1-z)^2-z^3,z)` and get

$$z = -\frac{\left(9\sqrt{23} + 11\sqrt{3}\right)^{\frac{2}{3}} \left(\sqrt{3}\,i + 1\right) + 5\,2^{\frac{2}{3}}\,3^{\frac{5}{6}}\,i - 2^{\frac{4}{3}}\,3^{\frac{1}{6}}\left(9\sqrt{23} + 11\sqrt{3}\right)^{\frac{1}{3}} - 5\,2^{\frac{2}{3}}\,3^{\frac{1}{3}}}{2^{\frac{4}{3}}\,3^{\frac{7}{6}}\left(9\sqrt{23} + 11\sqrt{3}\right)^{\frac{1}{3}}},$$

$$z = \frac{\left(9\sqrt{23} + 11\sqrt{3}\right)^{\frac{2}{3}} \left(\sqrt{3}\,i - 1\right) + 5\,2^{\frac{2}{3}}\,3^{\frac{5}{6}}\,i + 2^{\frac{4}{3}}\,3^{\frac{1}{6}}\left(9\sqrt{23} + 11\sqrt{3}\right)^{\frac{1}{3}} + 5\,2^{\frac{2}{3}}\,3^{\frac{1}{3}}}{2^{\frac{4}{3}}\,3^{\frac{7}{6}}\left(9\sqrt{23} + 11\sqrt{3}\right)^{\frac{1}{3}}},$$

$$z = \frac{\left(9\sqrt{23} + 11\sqrt{3}\right)^{\frac{2}{3}} + 2^{\frac{1}{3}}\,3^{\frac{1}{6}}\left(9\sqrt{23} + 11\sqrt{3}\right)^{\frac{1}{3}} - 5\,2^{\frac{2}{3}}\,3^{\frac{1}{3}}}{2^{\frac{1}{3}}\,3^{\frac{7}{6}}\left(9\sqrt{23} + 11\sqrt{3}\right)^{\frac{1}{3}}}$$

Wolfram Alpha even gives us a nice diagram

`http://www.wolframalpha.com/input/?i=(1-z)%5E2-z%5E3+%3D+0`

from which we see that we have a small real and two larger complex conjugated singularities. We evaluate them numerically and see that their magitude are $\beta \approx 1.32471$ and $\alpha \approx 0.5698402909980533$. The dominant singularity is α. We decomposed the denominator of $S(z)$ into three roots of degree one. The function $S(z)(z-\alpha)$ is therefore analytical at α. This means that α is a pole of first order.

For a meromorph generating function $B(z)$ with poles $\alpha_1, \ldots, \alpha_m$ we know that there exist polynomials $P_1(n), \ldots, P_m(n)$ such that

$$[z^n]B(z) = \sum_{j=1}^{m} P_j(n)\alpha_j^n$$

and the degree of the polynomial $P_j(n)$ is one smaller than the order of the pole α_j. Since we have three poles of first order the polynomials are constants. We have

$$s_n = c_1(\alpha^{-n}) + c_2(\beta^{-n}).$$

for some constants c_1 and c_2.

3. The exponential growth is $\alpha^{-n} \approx 1.754877666246692655^n$.

4. Notice that $c_2(\beta^{-n}) = O(0.8^n)$. If we can find the hidden factor c_1 we get a good approximation with vanishing additive error for s_n. We can decompose

$$S(z) = \sum_{n=1}^{\infty} c_1\alpha^{-n}z^n + c_2\beta^{-n}z^n = \frac{c_1}{1 - z/\alpha} + \frac{c_2}{1 - z/\beta}$$

Let

$$B(z) = \frac{1}{1 - z/\alpha} \quad \text{and} \quad E(z) = \frac{1}{1 - z/\beta}$$

Notice that $\lim_{z\to\alpha} B(z) = \infty$ while $\lim_{z\to\alpha} E(z)$ is constant. Then

$$\lim_{z\to\alpha} \frac{S(z)}{B(z)} = \lim_{z\to\alpha} \frac{c_1 B(z) + c_2 E(z)}{B(z)} = c_1.$$

We use maxima to approximate

$$\lim_{z\to\alpha} \frac{S(z)}{B(z)} = \lim_{z\to\alpha} \frac{z(1-z)(1-z/\alpha)}{(1-z)^2 - z^3} \approx 0.23448675.$$

We use the following programm to verify the correctness

```
s = range(0,1000)
a = range(0,1000)
s[0] = 0
s[1] = 1
a[0] = 0
a[1] = 0
for n in range(2, 100):
  s[n] = a[n-2]+s[n-1]
  a[n] = a[n-1]+s[n-1]
  print n, s[n], 1.0*s[n]/s[n-1], 0.23448675*1.754877666246692655**n/s[n]
```

The last line states

```
99 3552680714539332284392 41 1.75487766625 0.999999931817.
```

Indeed, after 100 iterations we only make a multiplicative error of 0.999999931817.

Problem H29 (10 credits)
In the lecture, we used the exponential generating function $I(z)$ for the number of involutions to demonstrate the power of the saddle point method. In this exercise, you should derive this EGF. Remember, an *involution* is a permutation which is self-inverse.
First, find the recurrence relation for I_n where I_n is the number of involutions over n elements. Then use the usual toolbox for EGF to find an ordinary differential equation for $I(z)$. You can solve this by using tools like Wolfram Alpha or use that $I(z) = ce^{f(z)}$.

Solution
Consider an involution σ over $[n]$ for $n \geq 2$. If n is a fixed point of σ, then $\sigma|_{[n-1]}$ is an involution over $[n-1]$. Otherwise, $\sigma(n) \neq n$ and $\sigma(\sigma(n)) = n$. Then $\sigma|_{[n-1]\setminus\{\sigma(n)\}}$ is an involution over $n-2$ elements. There are $n-1$ choices for $\sigma(n)$. As all these choices are mutually exclusives we get this recurrence relation (with the two base cases):

$$I_n = I_{n-1} + (n-1)I_{n-2} + (n=0) + (n=1).$$

We transform both sides:

$$\sum_{n=0}^{\infty} a_n \frac{z^n}{n!} = \sum_{n=0}^{\infty} a_{n-1} \frac{z^n}{n!} + \sum_{n=0}^{\infty} (n-1) a_{n-2} \frac{z^n}{n!} + 1 + z$$

Remember the following two rules for EGFs:

$$(1) \qquad \int_0^z A(t)dt = \sum_{n=0}^{\infty} a_{n-1} \frac{z^n}{n!}$$

$$(2) \qquad zA(z) = \sum_{n=0}^{\infty} n a_{n-1} \frac{z^n}{n!}$$

The first summand of the right side can be expressed by rule (1). The second summand can be expressed by first applying rule (1) and then rule (2).

$$A(z) = \int_0^z A(t)dt + \int_0^z \sum_n^{\infty} n a_{n-1} \frac{t^n}{n!} dt + 1 = \int_0^z A(t)dt + \int_0^z tA(t)dt + 1$$

We differentiate:

$$A'(z) = A(z) + zA(z)$$

We substitute $A(z) = e^{f(z)}$ and solve for $f(z)$.

$$f'(z)e^{f(z)} = e^{f(z)} + ze^{f(z)}$$

$$f'(z) = 1 + z$$

$$f(z) = z + z^2/2 + c$$

This means

$$A(z) = e^{z+z^2/2+c}.$$

For $z = 0$ we have

$$e^c = A(0) = \int_0^0 A(t)dt + \int_0^0 tA(t)dt + 1 = 1$$

which implies $c = 0$ and therefore

$$A(z) = e^{z+z^2/2}.$$