Analysis of Algorithms, WS 2020
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Exercise Sheet with solutions 11

Problem T25

Determine g_n up to an additive error of $O(4^n)$ for

$$G(z) = \sum_{n=0}^{\infty} g_n z^n = \frac{15z^2 + 8z + 1}{15z^2 - 8z + 1}.$$

Solution

We have

$$G(z) = \frac{15z^2 + 8z + 1}{(3z - 1)(5z - 1)},$$

the singularities are at $\frac{1}{3}$ and $\frac{1}{5}$, and both are poles of first order. Because of

$$G(z) \sim \frac{8}{5} \frac{1}{(z - \frac{1}{5})} \text{ for } z \to \frac{1}{5}$$

the difference between g_n and $[z^n]\frac{8}{1-5z}$ is at most $O(3^n)$. Therefore $g_n = 8 \cdot 5^n + O(3^n)$.

Problem T26

Let

$$U(z) := \frac{1 - z - \sqrt{(1 - 3z)(1 + z)}}{2z}.$$

Prove that $[z^n]U(z) = 3^n n^{O(1)}$ without doing any computations. Then find out what the constant in the monomial is, i.e., for what c is $[z^n]U(z) = \Theta(n^c 3^n)$.

Solution

The dominant singularity 1/3 is an algebraic singularity of order $c = 1/2$. Therefore $[z^n]U(z) = \Theta(n^{-c-1}3^n) = \Theta(3^n/n^{3/2})$.

Problem T27

In exercise H25 we established roughly the number of 2-3-trees. Now we want to go for a better estimate.

What kind of singularity is the dominant one in the corresponding generating function?

The following `maxima` output, which finds roots of equations, might help you to answer this question:

$solve(Q = 1 + z * Q^2 + z * Q^3, Q);$

$$\left[Q = \left(-\frac{\sqrt{3}i}{2} - \frac{1}{2}\right)\left(\frac{\sqrt{\frac{z^2+11z-1}{z}}}{3^{\frac{3}{2}}z} - \frac{2}{3z} - \frac{1}{27}\right)^{\frac{1}{3}} - \frac{\left(\frac{\sqrt{3}i}{2} - \frac{1}{2}\right)\left(-\frac{1}{3z} - \frac{1}{9}\right)}{\left(\frac{\sqrt{\frac{z^2+11z-1}{z}}}{3^{\frac{3}{2}}z} - \frac{2}{3z} - \frac{1}{27}\right)^{\frac{1}{3}}} - \frac{1}{3},\right.$$

$$Q = \left(\frac{\sqrt{3}i}{2} - \frac{1}{2}\right)\left(\frac{\sqrt{\frac{z^2+11z-1}{z}}}{3^{\frac{3}{2}}z} - \frac{2}{3z} - \frac{1}{27}\right)^{\frac{1}{3}} - \frac{\left(-\frac{\sqrt{3}i}{2} - \frac{1}{2}\right)\left(-\frac{1}{3z} - \frac{1}{9}\right)}{\left(\frac{\sqrt{\frac{z^2+11z-1}{z}}}{3^{\frac{3}{2}}z} - \frac{2}{3z} - \frac{1}{27}\right)^{\frac{1}{3}}} - \frac{1}{3},$$

$$\left.Q = \left(\frac{\sqrt{\frac{z^2+11z-1}{z}}}{3^{\frac{3}{2}}z} - \frac{2}{3z} - \frac{1}{27}\right)^{\frac{1}{3}} - \frac{-\frac{1}{3z} - \frac{1}{9}}{\left(\frac{\sqrt{\frac{z^2+11z-1}{z}}}{3^{\frac{3}{2}}z} - \frac{2}{3z} - \frac{1}{27}\right)^{\frac{1}{3}}} - \frac{1}{3}\right]$$

Solution

The first step is to find out, what the generating function is. There are three choices and only one can be correct. We can rule out the second and third one because they have singularities at the origin.

The dominant singularity is $\frac{1}{2}(5^{3/2} - 11)$. It is clearly not a pole, so the question remains whether it could be an algebraic singularity. If we look at functions of the form

$$(\sqrt{(}z - z_0)f(z) + g(z))^{1/3},$$

where $f(z)$ and $g(z)$ are analytical at z_0 we have to conclude that z_0 cannot be an algebraic singularity:

Multiplying by $(z - z_0)^c$ yields

$$((z - z_0)^{1/2+3c}f(z) + (z - z_0)^{3c}g(z))^{1/3}.$$

However we choose c we cannot make both $1/2 + 3c$ and $3c$ to be integers. This situation applies in our generating function, so the dominant singularity is not algebraic.

Problem H26 (5 credits)

$$A(z) = \frac{\sqrt{1 - z^7}}{2z^2 - 3z + 1} \qquad B(z) = \frac{1 - z^2}{e^{z+3z^2}} \qquad C(z) = z^5 + 3z^2(z^3 + z^2 + 8)$$

Order the coefficients of the sequences a_n, b_n, and c_n in increasing order by their asymptotic growth and give a proof.

Solution

$A(z)$ has a dominant singularity at $\frac{1}{2}$, which means it as an exponential growth of 2^n. $B(z)$ does not have any singularities and has therefore subexponential growth and $[z^n]C(z) = 4(n = 5) + 3(z = 4) + 24(z = 2)$ is always zero except for finite many exceptions. From that follows $C(z) < B(z) < A(z)$.

Problem H27 (10 credits)
We continue exercise T26 where

$$U(z) = \frac{1 - z - \sqrt{(1-3z)(1+z)}}{2z}.$$

and we found the constant c with $[z^n]U(z) = \Theta(n^c 3^n)$.
Now also find the multiplicative constant in the Θ-notation, i.e., find a simple function $f(n)$ such that $[z^n]U(z) \sim f(n)$.

Solution
We estimate $U(z)$ in the vicinity of the dominant singularity:

$$U(z) \sim \frac{1 - \frac{1}{3} - \sqrt{(1-3z)(1+\frac{1}{3})}}{2 \cdot \frac{1}{3}} = 1 - \sqrt{3}\sqrt{1-3z} \text{ for } z \to \frac{1}{3}$$

The theorem about algebraic singularities says that

$$[z^n]U(z) \sim -\frac{\sqrt{3}\, n^{-3/2}\, 3^n}{\Gamma(-1/2)} = \frac{\sqrt{3}\, n^{-3/2}\, 3^n}{2\sqrt{\pi}}.$$

Problem H28 (10 credits)
Approximate $[z^n]\frac{1}{2-e^z}$ up to an error of $O(12^{-n})$.

Solution
In the lecture, we found the first term (of the dominant singularity), namely $\frac{1}{2}(\frac{1}{\ln 2})^{n+1}$.
So we take a look at the singularity with the second highest absolute value, which is $\ln 2 \pm 2\pi i$.
Both are poles of order 1. Let us see how $S(z)$ behaves asymptotically for $z \to \ln 2 \pm 2\pi i$. We have that $e^z \sim 2(1 - \ln 2 \mp 2\pi i + z)$ for $z \to \ln 2 \pm 2\pi i$ and therefore

$$\begin{aligned}\frac{1}{2-e^z} &\sim \frac{1}{2 - (2 - 2\ln 2 \mp 4\pi i + 2z)} \\ &= \frac{1}{2}\frac{1}{\ln 2 \mp 2\pi i}\frac{1}{1 - \frac{z}{\ln 2 \mp 2\pi i}} \\ &= \frac{1}{2}\frac{1}{\ln 2 \mp 2\pi i}\sum_{n=0}^{\infty}(\frac{1}{\ln 2 \mp 2\pi i})^n z^n\end{aligned}$$

With the Theorem 9 we get:

$$\begin{aligned}[z^n]S(z) &= \frac{1}{2}\left((\frac{1}{\ln 2})^{n+1} + (\frac{1}{\ln 2 + 2\pi i})^{n+1} + (\frac{1}{\ln 2 - 2\pi i})^{n+1}\right) + O(r)^{-n} \\ &= \frac{1}{2}\left((\frac{1}{\ln 2})^{n+1} + r^{n+1}(e^{i\phi(n+1)} + e^{-i\phi(n+1)})\right) + O(r)^{-n} \\ &= \frac{1}{2}\left((\frac{1}{\ln 2})^{n+1} + r^{n+1} i2\cos(\phi(n+1))\right) + O(r)^{-n}\end{aligned}$$

with $r = 1/\sqrt{\ln^2 2 + 4\pi^2} \approx 12.58547409739904$, $\phi = \arctan(\frac{2\pi}{\ln 2})$.