Analysis of Algorithms, WS 2020
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Exercise Sheet 11

Problem T25

Determine g_n up to an additive error of $O(4^n)$ for

$$G(z) = \sum_{n=0}^{\infty} g_n z^n = \frac{15z^2 + 8z + 1}{15z^2 - 8z + 1}.$$

Problem T26

Let

$$U(z) := \frac{1 - z - \sqrt{(1-3z)(1+z)}}{2z}.$$

Prove that $[z^n]U(z) = 3^n n^{O(1)}$ without doing any computations. Then find out what the constant in the monomial is, i.e., for what c is $[z^n]U(z) = \Theta(n^c 3^n)$.

Problem T27

In exercise H25 we established roughly the number of 2-3-trees. Now we want to go for a better estimate.
What kind of singularity is the dominant one in the corresponding generating function?
The following `maxima` output, which finds roots of equations, might help you to answer this question:
$solve(Q = 1 + z * Q^2 + z * Q^3, Q)$;

$$\left[Q = \left(-\frac{\sqrt{3}i}{2} - \frac{1}{2}\right)\left(\frac{\sqrt{\frac{z^2+11z-1}{z}}}{3^{\frac{3}{2}}z} - \frac{2}{3z} - \frac{1}{27}\right)^{\frac{1}{3}} - \frac{\left(\frac{\sqrt{3}i}{2} - \frac{1}{2}\right)\left(-\frac{1}{3z} - \frac{1}{9}\right)}{\left(\frac{\sqrt{\frac{z^2+11z-1}{z}}}{3^{\frac{3}{2}}z} - \frac{2}{3z} - \frac{1}{27}\right)^{\frac{1}{3}}} - \frac{1}{3},\right.$$

$$Q = \left(\frac{\sqrt{3}i}{2} - \frac{1}{2}\right)\left(\frac{\sqrt{\frac{z^2+11z-1}{z}}}{3^{\frac{3}{2}}z} - \frac{2}{3z} - \frac{1}{27}\right)^{\frac{1}{3}} - \frac{\left(-\frac{\sqrt{3}i}{2} - \frac{1}{2}\right)\left(-\frac{1}{3z} - \frac{1}{9}\right)}{\left(\frac{\sqrt{\frac{z^2+11z-1}{z}}}{3^{\frac{3}{2}}z} - \frac{2}{3z} - \frac{1}{27}\right)^{\frac{1}{3}}} - \frac{1}{3},$$

$$\left.Q = \left(\frac{\sqrt{\frac{z^2+11z-1}{z}}}{3^{\frac{3}{2}}z} - \frac{2}{3z} - \frac{1}{27}\right)^{\frac{1}{3}} - \frac{-\frac{1}{3z} - \frac{1}{9}}{\left(\frac{\sqrt{\frac{z^2+11z-1}{z}}}{3^{\frac{3}{2}}z} - \frac{2}{3z} - \frac{1}{27}\right)^{\frac{1}{3}}} - \frac{1}{3}\right]$$

Problem H26 (5 credits)

$$A(z) = \frac{\sqrt{1-z^7}}{2z^2 - 3z + 1} \qquad B(z) = \frac{1-z^2}{e^{z+3z^2}} \qquad C(z) = z^5 + 3z^2(z^3 + z^2 + 8)$$

Order the coefficients of the sequences a_n, b_n, and c_n in increasing order by their asymptotic growth and give a proof.

Problem H27 (10 credits)
We continue exercise T26 where

$$U(z) = \frac{1 - z - \sqrt{(1-3z)(1+z)}}{2z}.$$

and we found the constant c with $[z^]U(z) = \Theta(n^c 3^n)$.
Now also find the multiplicative constant in the Θ-notation, i.e., find a simple function $f(n)$ such that $[z^n]U(z) \sim f(n)$.

Problem H28 (10 credits)
Approximate $[z^n]\frac{1}{2-e^z}$ up to an error of $O(12^{-n})$.