Analysis of Algorithms, WS 2020
Prof. Dr. P. Rossmanith
Dr. E. Burjons, H. Lotze, D. Mock

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Exercise Sheet with solutions 10

Problem T22

Find the exponential growth of the following functions:

a) $2^n n^3$

b) $\left(\frac{2}{3}\right)^{2n} + 5$

c) $[z^n]\frac{1}{\sqrt{1-5z}}$

d) $[z^n]\frac{z^2-1}{(z-1)(z-5)}$

e) $[z^n]\frac{1}{e-e^{3/2-z^2}}$

Solution

a) 2^n

b) 1

c) 5^n

d) $(1/5)^n$

e) $\sqrt{2}^n$

We only show two solutions as a) is kind of easy and c) - e) are very similar.

b) Two show that $\left(\frac{2}{3}\right)^{2n} + 5 \asymp 1$. We check the first condition of the alternative definition: $\left(\frac{2}{3}\right)^{2n} + 5 > (1-\varepsilon)^n$ as $(1-\varepsilon)^n < 5$ for every $n > 0$. The second condition $\left(\frac{2}{3}\right)^{2n} + 5 < (1+\varepsilon)^n$ holds once $(1+\varepsilon)^n$ reaches 6 for every subsequent n. This happens as the function is strictly monotone increasing.

d) We use the theorem linking the exponential growth to the dominant singularity of its generating function. Every singularity of $\frac{z^2-1}{(z-1)(z-5)}$ has to fulfill $(z-1)(z-5) = 0$ which holds for $z = 1$ and $z = 5$. However, $z = 1$ is *not* a singularity as the function can be continued at this undefined point (the limit at this points exits). But 5 is a singularity. Because it is the only one, it is the dominant singularity. Hence $\frac{z^2-1}{(z-1)(z-5)} \asymp (1/5)^n$.

Note that the generating function in e) has many singularities on the complex plane but the dominant singularity (of power series with non-negative coefficients) is always real.

Problem T23

Find very large and very small functions with exponential growth 1, 0 and ∞.

Solution

exponential growth 1:

- $(1/1000)^{n/\log(n)}$
- $(1/n)^3$
- n^3
- $1000^{n/\log(n)}$
- $(1/2)^{\sqrt{n}}$
- 1
- $2^{\sqrt{n}}$

exponential growth 0:

- 0
- $1/n!$
- $(1/2)^{n^2}$
- $(0.999)^{n\log(n)}$

exponential growth ∞:

- $(1.001)^{n\log(n)}$
- 2^{2^n}

Problem T24

Sort the following generating functions *within one minute* by their exponential growth!

1. $A(z) = \frac{1}{\sqrt{1-z/2}}$
2. $B(z) = \frac{1}{1-e^{z-1/3}}$
3. $C(z) = \frac{1+z}{1-z}$

Solution

We only need to sort them by the absolute value of the dominant singularities. $A_n \asymp 1/2^n$, $B_n \asymp 3^n$, $C_n \asymp 1$. Therefore $A_n \leq C_n \leq B_n$.

Problem H23 (5 credits)

Prove that

$$\binom{-r}{n} = (-1)^n \binom{r+n-1}{n}$$

for $r \in \mathbf{R}, n \in \mathbf{Z}$.

Solution

We have

$$\begin{aligned}
\binom{-w}{n} &= \frac{(-w)^{\underline{n}}}{n!} \\
&= \frac{(-w)(-w-1)\cdots(-w-n+1)}{n!} \\
&= \frac{(-1)^n w(w+1)\cdots(w+n-1)}{n!} \\
&= \frac{(-1)^n (n+w-1)^{\underline{n}}}{n!} \\
&= (-1)^n \binom{n+w-1}{n}
\end{aligned}$$

Problem H24 (10 credits)

Prove that

$$[z^n](1-z)^w \sim \frac{n^{-w-1}}{\Gamma(-w)}$$

for $w \in \mathbf{C}$ without using the theorem of the lecture. (The idea of this assignment is to get a deeper insight into the theorem.)

Hint: Use Newton's formula, then the first exercise on this sheet. Now replace the binomial coefficient by factorials or the gamma function. In the first case, you need to be careful with a definition of factorials for real numbers. In general, however, $\Gamma(n+1) = n!$.

Solution

$$
\begin{aligned}
[z^n](1-z)^w &= \binom{w}{n}(-1)^n \\
&= \binom{n-w-1}{n} \\
&= \frac{(n-w-1)!}{n!(-w-1)!} \\
&= \frac{1}{n^{\underline{w+1}}(-w-1)!} \\
&= \frac{1}{n^{\underline{w+1}}\Gamma(-w)} \\
&= \frac{n^{-w-1}}{\Gamma(-w)}\left(1+O\left(\frac{1}{n}\right)\right)
\end{aligned}
$$

Problem H25 (10 credits)

In this exercise we will look at 2-3-trees. They are rooted, ordered trees. Each internal node has either two or three children. As usual, the size of a 2-3-tree will be the number of its internal nodes.

1. How can you define 2-3-trees recursively?
2. Enumerate all 2-3-trees of size two. How many are there? How many trees exist of sizes zero and one?
3. Find a generating function $Q(z)$ for the number q_n of 2-3-trees with size n.
4. What is the dominant singularity of $Q(z)$ and what is the exponential growth of q_n? Use a computer algebra system. Do not give up when you see horrifying formulas.

Solution

We get immeadiately the equation $Q(z) = zQ(z)^2 + zQ(z)^3 + 1$. If we solve it with the help of a computer algebra system, we get a very messy result, but we can spot the expression $\sqrt{z^2+11z-1}$, which defines one the singularities. It seems that it is probably the dominant one. If that is true, then $\alpha = (5^{3/2} - 11)/2$ is the singularity for which we are looking. The exponential growth is then $\alpha^{-n} \approx 11.09016994374933^n$.

A closer look shows that this is indeed the dominant singularity.