**Analysis of Algorithms, WS 2020**
Prof. Dr. P. Rossmanith
Dr. E. Burjons, H. Lotze, D. Mock

Date: 13 January 2021

# Exercise Sheet with solutions 09

If $\int_1^n |f^{(i)}(x)|\,dx$ exists for $1 \le i \le 2m$, then

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,dx + \frac{1}{2}f(n) + C + \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(n) + R_m,$$

where $R_m = O\left(\int_1^n |f^{(2m)}(x)|\,dx\right)$ and $B_k = n![z^n]z/(e^z-1)$ are the Bernoulli-numbers:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $B_n$ | 1 | $-\frac{1}{2}$ | $\frac{1}{6}$ | 0 | $-\frac{1}{30}$ | 0 | $\frac{1}{42}$ |

## Problem T20

Approximate the following sum up to an error of $O(n^{-5})$:

$$\sum_{k=1}^{n} \frac{1}{k^2}$$

Find the constant $C$ in Euler's summation formula by looking up $\sum_{k=1}^{\infty} \frac{1}{k^2}$. Test your result for $n = 1000$. Use your favorite computing software.

## Solution

Euler's summation formula gives us

$$\sum_{k=1}^{n} \frac{1}{k^2} = \int_1^n \frac{dx}{x^2} + \frac{1}{2n^2} + C + \frac{B_2}{2}\left(-\frac{2}{n^3}\right) + R_2.$$

Moreover, it holds that

$$R_2 = O\Big(\int_n^\infty \Big| \left(\frac{1}{x^2}\right)^{(4)} \Big|_{x=n} \Big| \, dx\Big) = O(n^{-5}),$$

and

$$\sum_{k=1}^{n} \frac{1}{k^2} = \frac{\pi^2}{6} - \frac{1}{n} + \frac{1}{2n^2} - \frac{1}{6n^3} + O(n^{-5})$$

because of this identity

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \zeta(2) = \frac{\pi^2}{6}.$$

The value for $n = 1000$ is $1.64393456668156\ldots$. We get these results

| | |
|---|---|
| $\frac{\pi^2}{6}$ | $\underline{1.64}49\ldots$ |
| $\frac{\pi^2}{6} - \frac{1}{n}$ | $\underline{1.6439340}\ldots$ |
| $\frac{\pi^2}{6} - \frac{1}{n} + \frac{1}{2n^2}$ | $\underline{1.643934566}8\ldots$ |
| $\frac{\pi^2}{6} - \frac{1}{n} + \frac{1}{2n^2} - \frac{1}{6n^3}$ | $\underline{1.64393456668156}\ldots$ |

**Problem T21**

If you use Euler summation on a polynomial function, can you get an *exact* solution? Prove it or find a counterexample.

**Solution**

It is indeed possible to get an exact solution using Euler summation on a polynomial function. If we take a look at the Euler summation formula it states that:
If $\int_1^n |f^{(i)}(x)|\, dx$ exists for $1 \leq i \leq 2m$, then

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\, dx + \frac{1}{2} f(n) + C + \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(n) + R_m,$$

where $R_m = O\left(\int_1^n |f^{(2m)}(x)|\, dx\right)$ and $B_k = n![z^n]z/(e^z - 1)$ are the Bernoulli-numbers.
We see that we have an error term that depends on the $2m$-th derivative of $f$. If we think about the derivatives of polynomials we quickly notice that if we derive a polynomial of degree $t$, $t+1$ times, the derivative becomes 0, making the error term disappear if we choose $m$ in such a way that for a polynomial of degree $t$, $t \leq 2m$. So, we obtain an exact formula.
Moreover, the other non-exact term we have is the additive constant, which we can obtain by substituting for instance for $n = 1$ on that particular polynomial but it is even easier than that. If we realize that both sides of the equation are polynomials we can even substitute for $n = 0$. This makes the left hand side 0 thus we realize that the constant term of the polynomial on the right hand side must be 0. This does not mean that the constant $C$ is 0 but that when we add $C$ to the other constants that appear when applying the formula we get 0.

**Problem H21** (10 credits)

Approximate the following sum up to an error of $O(n^{-5})$:

$$\sum_{k=1}^{n} \frac{1}{k^{5/2}}$$

**Solution**

Euler's summation formula gives us

$$\sum_{k=1}^{n} \frac{1}{k^{5/2}} = \int_1^n \frac{1}{k^{5/2}} dx + \frac{1}{2n^{5/2}} + C + \frac{B_2}{2}\left(-\frac{5/2}{n^{7/2}}\right) + R_2.$$

Moreover, it holds that

$$R_2 = O\Big(\int_n^\infty \Big| \left(\frac{1}{x^{5/2}}\right)^{(4)} \Big|_{x=n} \Big|\, dx\Big) = O(n^{-5}),$$

and

$$\sum_{k=1}^{n} \frac{1}{k^2} = 1.3419 - \frac{2}{3n^{3/2}} + \frac{1}{2n^{5/2}} - \frac{5}{24n^{7/2}} + O(n^{-5})$$

because of this identity

$$\sum_{k=1}^{\infty} \frac{1}{k^{5/2}} = \zeta(2.5) = 1.3419...$$

**Problem H22** (10 credits)

Find a function $f(n)$ in closed form such that

$$\prod_{k=1}^{n} k^k = f(n)(1 + O(1/n^2)).$$

Use Euler summation. It is okay if you cannot find the correct constant in the sum.

**Solution**

Similar to $n!$ we turn to logarithms in order to turn the product into a sum and get immeadiately (with $f(x) = x \ln x$)

$$\ln\Big(\prod_{k=1}^{n} k^k\Big) = \sum_{k=1}^{n} k \ln k \sim \int_1^n x \ln x \, dx + \frac{n \ln n}{2} + C + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(n).$$

We need to find out what the integral and what the derivatives of $f$ are. We get $\int x \ln x \, dx = x^2(2 \ln x - 1)/4 + C$ and $f'(x) = ln(x) + 1$, $f''(x) = 1/x$, and $f^{(k)}(x) = (-1)^k (k-2)! x^{-k+1}$ for $k \geq 2$. This gives us

$$\sum_{k=1}^{n} k \ln k \sim \frac{n^2(2 \ln n - 1)}{4} + \frac{n \ln n}{2} + C' + \frac{1/6}{2!} \ln(n) + \sum_{k=2}^{\infty} \frac{B_{2k}(-1)^{2k-1}(2k-3)!}{(2k)! n^{2k-2}}$$

$$= \sum_{k=1}^{n} k \ln k = \frac{n^2 \ln n}{2} - \frac{n^2}{4} + \frac{n \ln n}{2} + \frac{\ln n}{12} + C' - \frac{1}{720 n^2} - \frac{1}{5040 n^4} \pm \cdots$$

Now that we have an asymptotic expansion of the logarithm of the desired product we just have to apply the exponential function to it and we get:

$$\bar{\sigma} n^{n^2/2 - n^2/4 \ln n + n/2 + 1/12} (1 + O(1/n^2))$$

Unfortunately, we do not know the constant $\bar{\sigma}$ yet...