Exercise for Analysis of Algorithms

Exercise E1

Determine a recurrence relation for the number B_n of comparisons, when the following algorithm is used to find an element x contained in an array $a[1], \ldots, a[n]$. We assume that the array is sorted in increasing order and contains x.
Then compute B_3.

Algorithm: Binary Search with randomly chosen pivot element

1. Choose randomly and with uniform probability an $i \in \{1, \ldots, n\}$.
2. If $a[i] = x$, output i and halt.
3. Continue recursively on the left ($x < a[i]$) or right ($x > a[i]$) subarray.

Exercise E2

$$A(z) = \frac{\sqrt{1-z^7}}{2z^2-3z+1} \qquad B(z) = \frac{1-z^2}{e^{z+3z^2}} \qquad C(z) = z^5 + 3z^2(z^3+z^2+8)$$

Order the coefficients of the sequences a_n, b_n, and c_n in increasing order by their asymptotic growth and give a proof.

Exercise E3

Consider the number B_n of $2-3$–trees (each inner node has exactly two or three children) with n leaves. Does B_n grow asymptotically slower or faster than 5^n?
Hint: The following `maxima` output, which finds roots of equations, might help you to answer this question: $solve(T^3 + T^2 - T + z = 0, T)$:

$$\Bigg[T = \left(-\frac{\sqrt{3}\,i}{2} - \frac{1}{2}\right)\left(\frac{\sqrt{27\,z^2+22\,z-5}}{6\,\sqrt{3}} - \frac{27\,z+11}{54}\right)^{\frac{1}{3}} + \frac{4\left(\frac{\sqrt{3}\,i}{2}-\frac{1}{2}\right)}{9\left(\frac{\sqrt{27\,z^2+22\,z-5}}{6\,\sqrt{3}} - \frac{27\,z+11}{54}\right)^{\frac{1}{3}}} - \frac{1}{3},$$

$$T = \left(\frac{\sqrt{3}\,i}{2} - \frac{1}{2}\right)\left(\frac{\sqrt{27\,z^2+22\,z-5}}{6\,\sqrt{3}} - \frac{27\,z+11}{54}\right)^{\frac{1}{3}} + \frac{4\left(-\frac{\sqrt{3}\,i}{2}-\frac{1}{2}\right)}{9\left(\frac{\sqrt{27\,z^2+22\,z-5}}{6\,\sqrt{3}} - \frac{27\,z+11}{54}\right)^{\frac{1}{3}}} - \frac{1}{3},$$

$$T = \left(\frac{\sqrt{27\,z^2+22\,z-5}}{6\,\sqrt{3}} - \frac{27\,z+11}{54}\right)^{\frac{1}{3}} + \frac{4}{9\left(\frac{\sqrt{27\,z^2+22\,z-5}}{6\,\sqrt{3}} - \frac{27\,z+11}{54}\right)^{\frac{1}{3}}} - \frac{1}{3}\Bigg]$$

Exercise E4

Determine g_n up to an additive error of $O(4^n)$ for

$$G(z) = \sum_{n=0}^{\infty} g_n z^n = \frac{15z^2+8z+1}{15z^2-8z+1}.$$