RWTH Aachen **WS 2017**
Theoretical Computer Science Exercise 8
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Exercise for Analysis of Algorithms

Exercise T16

Find the exponential growth of the following functions:

a) $2^n n^3$

b) $\left(\frac{2}{3}\right)^{2n} + 5$

c) $[z^n]\frac{1}{\sqrt{1-5z}}$

d) $[z^n]\frac{z^2-1}{(z-1)(z-5)}$

e) $[z^n]\frac{1}{e-e^{3/2-z^2}}$

Solution:

a) 2^n

b) $(4/9)^n$

c) 5^n

d) $(1/5)^n$

e) $\sqrt{2}^n$

Exercise T17

Find very large and very small functions with exponential growth 1, 0 and ∞.

Solution:
exponential growth 1:

- $(1/1000)^{n/\log(n)}$
- $(1/2)^{\sqrt{n}}$
- $(1/n)^3$
- 1
- n^3
- $2^{\sqrt{n}}$
- $1000^{n/\log(n)}$

exponential growth 0:

- 0
- $1/n!$
- $(1/2)^{n^2}$
- $(0.999)^{n\log(n)}$

exponential growth ∞:

- $(1.001)^{n\log(n)}$
- 2^{2^n}

Exercise T18

Sort the following generating functions *within one minute* by their exponential growth!

1. $A(z) = \frac{1}{\sqrt{1-z/2}}$
2. $B(z) = \frac{1}{1-e^{z-1/3}}$
3. $C(z) = \frac{(1+z)}{(1-z)}$

Solution:
We only need to sort them by the absolute value of the dominant singularities. $A_n \asymp 1/2^n$, $B_n \asymp 3^n$, $C_n \asymp 1$. Therefore $A_n \leq C_n \leq B_n$.

Exercise T19

Santa Claus wants to build a new landing strip for his reindeer. He has the following 1×2 tile he can use and rotate by 90 degrees: ▯. He wants to pave a strip of $2 \times n$. Since his contractor delivers the tiles in a specific color pattern he does not know how he can pave this strip so that it looks best. He wrote a computer program that enumerates all possible ways to pave the strip and then assigns it a beauty-value based on an evaluation function. Afterwards it outputs the best looking option. The evaluation takes $O(n)$ time. He thinks that, if the exponential growth of the running time is less than 4.5^n he can find the most beautiful landing strip in time for Christmas. Can he find it or is Christmas doomed?

Solution:

Let A be the number of configurations for the landing strip. We use the symbolic method to find a generating function.

$$A = | + ▯A + ⊟A$$

The generating function is thus

$$A(z) = 1 + zA(z) + z^2 A(z)$$
$$A(z) = \frac{1}{1 - z - z^2}$$

To estimate the exponential growth we have to find the root of this function.

$$z^2 + z - 1 = 0$$

Note that we have to find the smallest positive root.

$$z_{1,2} = -\frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + 1}$$
$$z_1 = -\frac{1}{2} + \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2} - \frac{1}{2} \approx 0.61803$$

The exponential growth is thus given by $(\frac{1}{z_1})^n = 1.61804^n$ and means Santa can find the best configuration in time for Christmas.

Exercise H12

Rudolph contracted another company for delivering the tiles, as Santa had originally in mind (see T19), and they delivered an additional 1×1 tile: ▫. This increases the options and ultimately the running time for Santa's program. Can he still find the best looking option in time if he also considers the new tile?

Solution:

Let us now count the area and not the length of a tile, that is $\square$ for example has now weight 2 and no longer 1. We have A for the configurations for a strip as before

$$A = | + {}^{\square}B + {}_{\square}B + \square A + \boxminus A$$

and B for configurations that are missing a $\square$ tile either in the top left or bottom left (the two cases are symmetric).

$$B = {}^{\square}A + {}^{\sqsupset}B$$

Which gives rise to the following generating functions

$$A(z) = 1 + 2zB(z) + z^2A(z) + z^4A(z)$$
$$B(z) = zA(z) + z^2B(z)$$

We rewrite the lower one as

$$B(z) = \frac{zA(z)}{1 - z^2}$$

And substitute it in the first one

$$A(z) = 1 + \frac{2z^2A(z)}{1 - z^2} + z^2A(z) + z^4A(z)$$
$$A(z) = \frac{1}{1 - z^2 - z^4 - \frac{2z^2}{1-z^2}}$$
$$A(z) = \frac{1 - z^2}{(1 - z^2)(1 - z^2 - z^4) - 2z^2}$$

We solve

$$(1 - z^2)(1 - z^2 - z^4) - 2z^2 = 0$$

and get as dominant singularity $\alpha \approx 0.504$. Since the area is $2n$ we get an exponential growth of $(\frac{1}{\alpha})^{2n} = (\frac{1}{0.504})^{2n} = 3.9366^n$ for the number of tilings for an $n \times 2$ landing strip, which luckily is still enough to find it in time for Christmas.

Exercise H13

In the lecture the following recurrence was given

$$a_{n+2} - (n + 2)a_{n+1} + na_n = n$$

What is the dimension of the solution space? Give a closed form expression for the starting conditions $a_0 = a_1 = 1$.

Solution:

In operator notation this recurrence relation looks as follows:

$$(E^2 - (n + 2)E + n)a_n = n$$

We can indeed factor this polynomial because $(E-1)(E-n) = E^2 - (n+2)E + n$. Please note that $En = (n + 1)E$. The recurrence relation has now the form

$$(E - 1)(E - n)a_n = n$$

and we start by solving $(E-1)b_n = n$ or, equivalently $b_{n+1} = b_n + n$. This is a hidden sum and we get the solution with the help of a theorem from the lecture. This results in

$$b_n = \sum_{k=0}^{n}(n-1) = \frac{n(n-1)}{2} + b_0 = \frac{n(n-1)}{2} + a_1.$$

In the final step we have to solve $(E-n)a_n = b_n = n(n-1)/2 - a_1$, which is a recurrence relation that can also be written as

$$a_{n+1} = na_n + \frac{n(n-1)}{2} + a_1. \tag{1}$$

This is a linear recurrence of first order. Solving it yields

$$a_n = \frac{(n-1)!}{2}\left(\sum_{k=1}^{n-1}\frac{k^2-k+2a_1}{k!}\right) + a_1(n-1)!. \tag{2}$$

As usual the result is in the form of a summation. Let us take a closer look at the interesting part inside the big parentheses:

$$\begin{aligned}
\sum_{k=1}^{n-1}\frac{k(k-1)+2a_1}{k!} &= \sum_{k=2}^{n-1}\frac{1}{(k-2)!} + \sum_{k=1}^{\infty}\frac{2a_1}{k!} = \sum_{k=0}^{n-3}\frac{1}{k!} + 2a_1\sum_{k=1}^{n-1}\frac{1}{k!} \\
&= \sum_{k=0}^{n-3}\frac{1}{k!} + 2a_1\Big(\sum_{k=0}^{n-1}\frac{1}{k!} - 1\Big) = e - \sum_{k=n-2}^{\infty}\frac{1}{k!} + 2a_1\Big(e - \sum_{k=n}^{\infty}\frac{1}{k!} - 1\Big) \\
&= e - O(1/(n-2)!) + 2a_1(e - O(1/n!) - 1) = 2a_1(e-1) + e + O(1/(n-2)!)
\end{aligned}$$

Inserting the result into (2) gives us an asymptotic estimate of a_n:

$$\begin{aligned}
a_n = \frac{(n-1)!}{2}\Big(2a_1(e-1) + e + O(1/(n-2)!)\Big) + a_1(n-1)! \\
= (n-1)!(a_1 e + e/2) + O(n)
\end{aligned}$$

But we want an exact solution so we need to be a bit more precise in the long calculation above:

$$\begin{aligned}
e - \sum_{k=n-2}^{\infty}\frac{1}{k!} + 2a_1\Big(e - \sum_{k=n}^{\infty}\frac{1}{k!} - 1\Big) = \\
e - \frac{1}{(n-2)!} - \frac{1}{(n-1)!} - \sum_{k=n}^{\infty}\frac{1}{k!} + 2a_1\Big(e - \sum_{k=n}^{\infty}\frac{1}{k!} - 1\Big) = \\
e - \frac{1}{(n-2)!} - \frac{1}{(n-1)!} - 2a_1(e-1) + O(1/n!)
\end{aligned}$$

Inserting the better estimate into (2) gives us a better asymptotic estimate:

$$\begin{aligned}
a_n = \frac{(n-1)!}{2}\Big(2a_1(e-1) + e - \frac{1}{(n-2)!} - \frac{1}{(n-1)!} + O(1/n!)\Big) + a_1(n-1)! \\
= (n-1)!(a_1 e + e/2) - n + O(1/n)
\end{aligned}$$

In fact, it is easy to see that our estimate is always a bit too high. Rounding down to the next integer gives the correct result for $n \geq 3$ for the starting condition $a_1 = 1$. Hence, we have a solution to the recurrence in closed form!

$$a_n = \lfloor (n-1)! \cdot e(a_1 + \frac{1}{2}) - n \rfloor \quad , \text{for } n \geq 3$$

The Solution dimension is one.