RWTH Aachen **WS 2017**
Theoretical Computer Science Exercise 5
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Exercise for Analysis of Algorithms

Exercise T9
Let $x \in \mathbf{R}_{\geq 0}$. Is $\lceil \sqrt{x} \rceil = \left\lceil \sqrt{\lceil x \rceil} \right\rceil$?

Solution:
Let $i \in \mathbf{N}$, s.t. $i^2 < x \leq (i+1)^2$. Then

$$i^2 < x \leq \lceil x \rceil \leq (i+1)^2.$$

Using monotonicity of $\sqrt{\cdot}$ we get

$$i = \sqrt{i^2} < \sqrt{x} \leq \sqrt{\lceil x \rceil} \leq \sqrt{(i+1)^2} = i+1$$

This means that both $\sqrt{x}$ and $\sqrt{\lceil x \rceil}$ are strictly greater than i and smaller than $i+1$. Since $i \in \mathbf{N}$ we can conclude that

$$\lceil \sqrt{x} \rceil = \left\lceil \sqrt{\lceil x \rceil} \right\rceil = i+1.$$

Exercise T10
How often is the loop in the following excerpt executed if $0 < i$ holds at the beginning?

```
while i <= j
   i := i+j;
   j:=j+10;
```

Solution:
We denote the value of i in the nth repetition by i_n (and similar for j_n). For $i_0 > j_0$, the while-loop is never executed. Let thus $0 < i_0 \leq j_0$. We obtain the recursion

$$\begin{aligned} i_n &= i_{n-1} + j_{n-1} \\ j_n &= j_{n-1} + 10 \end{aligned}$$

which yields (by insertion)

$$\begin{aligned} j_n &= j_0 + 10n \\ i_n &= i_{n-1} + 10(n-1) + j_0 \\ &= i_0 + \sum_{k=1}^{n} (10(k-1) + j_0) \\ &= i_0 + 5n(n-1) + nj_0. \end{aligned}$$

The loop is executed as long as $i_n - j_n \leq 0$, which implies

$$5n^2 + (j_0 - 15)n + i_0 - j_0 \leq 0.$$

We know that for a polynomial of degree two holds

$$ax^2 + bx + c = 0 \iff x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

For positive n we therefore need

$$n \leq \frac{15 - j_0 + \sqrt{(j_0 - 15)^2 - 20(i_0 - j_0)}}{10} =: a(i_0, j_0)$$

holds. In this case, the loop is hence executed $\lfloor a(i_0, j_0) \rfloor + 1$ times.

Exercise T11

Consider the following algorithm that searches an element x in a sorted array a of length $n = km + 1$:

```
i:= 1;
while a[i]<=x
  if a[i]=x then return i;
  i:=i+m;
  if i>n return 0;
for j=i-1 downto max(1,i-(m-1))
  if a[j]=x then return j;
  if a[j]<x then return 0;
return 0;
```

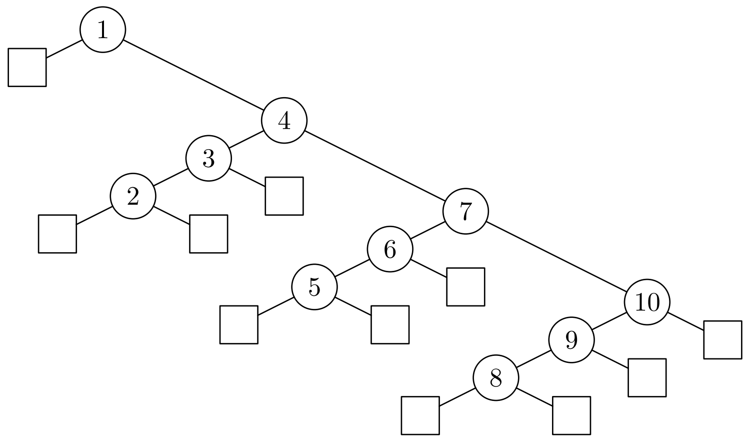
a) Draw the search tree and compute the internal and external path length for $n = 10$ and $m = 3$.

b) Determine C^+ and C^- for arbitrary m, k.

c) What is, for given n, the best choice for m w.r.t. the running time?

Solution:

a) The path lengths are $\pi(T) = 0 + 1 + 2 + 2 + 3 + 3 + 3 + 4 + 4 + 5 = 27$ and $\xi(T) = 1 + 3 + 4 + 4 + 4 + 4 + 5 + 5 + 5 + 6 + 6 = 47$:



b) It is sufficient to compute π. We then obtain

$$C^+ = \frac{\pi(T)}{n} + 1 \text{ and } C^- = \frac{\pi(T) + 2n}{n+1}.$$

Thus,

$$\begin{aligned}
\pi(T) &= \sum_{i=1}^{k} \sum_{j=i}^{i+m-1} j \\
&= \sum_{i=1}^{k} \frac{(i+m)(i+m-1)}{2} - \frac{i(i-1)}{2} \\
&= \frac{1}{2} \sum_{i=1}^{k} (2mi + m(m-1)) \\
&= \frac{mk(k+1) + km(m-1)}{2} \\
&= \frac{km^2 + mk^2}{2}.
\end{aligned}$$

Testing this for the example a) yields: $\pi(T) = (3 \cdot 9 + 3 \cdot 9)/2 = 27$.

c) In both cases, the search depths (in the average case) depends linear on $\pi(T)$. Hence we need to minimize this value. We express π using $n' := n - 1 = km$, thereby ignoring the constant.

$$\frac{km^2 + mk^2}{2} = \frac{n'^2/k + n'k}{2}$$

Deriving by k yields $n' - n'^2/k^2 = 0$. Hence, we have $k = \sqrt{n'}$. If $n = k^2 + 1$, this is the optimal value. Otherwise, we simply round to the closest integer, which is optimal because of symmetry.

Exercise H7

Use summation factors to solve the following recurrence:

$$\begin{aligned}
a_0 &= 0 \\
a_n &= \frac{a_{n-1}}{n} + \frac{1}{(n-1)!} \qquad \text{for } n \geq 1
\end{aligned}$$

Solution:

Plugging $y_n = 1/(n-1)!$ and $x_n = 1/n$ into the formula known from the lecture yields:

$$\begin{aligned}
a_n &= \frac{1}{(n-1)!} + \sum_{j=1}^{n-1} \frac{1}{(j-1)!} \frac{1}{j+1} \cdots \frac{1}{n} \\
&= \frac{1}{(n-1)!} + \sum_{j=1}^{n-1} \frac{j}{n!} \\
&= \frac{1}{(n-1)!} + \frac{1}{n!} \frac{n(n-1)}{2}
\end{aligned}$$

Exercise H8
Use the repertoire method to find a closed form for the following recurrence:

$$\begin{aligned} a_0 &= 5 \\ a_1 &= 9 \\ a_n &= na_{n-1} + n^2 a_{n-2} - n^4 - 3n^2 + 5 \qquad \text{for } n \geq 2 \end{aligned}$$

Solution:
We have $f(n) = a_n - na_{n-1} - n^2 a_{n-2} = -n^4 - 3n^2 + 5$. We guess different values for a_n and write down the resulting value for $f(n)$ into the table below.

a_n	$f(n)$	a_0	a_1
1	$-n^2 - n + 1$	1	1
n	$-n^3 + n^2 + 2n$	0	1
n^2	$-n^4 + 3n^3 - n^2 - n$	0	1

Let Z_i for $i = 1, 2, 3$ be the solutions of the first, second, and third line, respectively. Then $f(n) = 5Z_1 + 3Z_2 + Z_3$. For these, a_0 and a_1 are correct, and thus $a_n = 5 \cdot 1 + 3n + n^2$.