Analysis of Algorithms

Recall from class that if a complex-valued function $f(z)$ is analytic at 0 and R is the absolute value of the singularity nearest to the origin then the coefficient $f_n = [z^n]f(z)$ satisfies:

$$f_n \asymp \left(\frac{1}{R}\right)^n.$$

Exercise 11-1

Sort the following generating functions *within one minute* by their exponential growth!

1. $A(z) = 1/\sqrt{1 - z/2}$

2. $B(z) = (1 + z)/(1 - z)$

3. $C(z) = \frac{1}{1 - e^{-z - 1/3}}$

Solution: We only need to sort them by the absolute value of the dominant singularities. $A_n \asymp 1/2^n$, $B_n \asymp 1$, $C_n \asymp 3^n$.

Exercise 11-2

An algorithm is given an array of length $n \geq 0$ and, if $n \geq 2$, for each $1 \leq k \leq n$ calls itself on some random subarray of length k with probability $\frac{1}{2}$. Compute the exponential growth of the running time of this algorithm.

Solution:

Let t_n be the expected number of calls of the algorithm for an array of length n. Then $t_0 = t_1 = 1$ and $t_n = 1 + \frac{1}{2}\sum_{k=1}^{n} t_k$ for $n > 1$. To get an equation that holds for all n, we let

$$t_n = 1 + \frac{1}{2}\sum_{k=1}^{n} t_k - \frac{1}{2}(n = 1).$$

We make yet another modification so that in the equation above so that we may sum from $k = 0$:

$$t_n = 1 + \frac{1}{2}\sum_{k=0}^{n} t_k - \frac{1}{2}(n = 1) - \frac{1}{2}.$$

Define $T(z) = \sum_{n=0}^{\infty} t_n z^n$. Multiplying both sides by z^n and summing over n, we obtain:

$$T(z) = \frac{1}{1 - z} + \frac{1}{2(1 - z)}T(z) - \frac{1}{2}z + \frac{1}{2(1 - z)}$$

which gives the following functional relation: $T(z) = \frac{z^2 - z + 3}{1 - 2z}$. Since the dominant singularity is located at $z = \frac{1}{2}$, we get an asympotic running time of $\asymp 2^n$.

Exercise 11-3

Determine $[z^n]\frac{1}{2-e^z}$ up to an additive error of $O(12^{-n})$!

Solution: We first determine the dominant singularity, which is $\ln 2$. Since, by L'Hôpital,

$$\lim_{z \to \ln 2} \frac{\ln 2 - z}{2 - e^z} = \frac{1}{2},$$

we have

$$\frac{\ln 2 - z}{\ln 2 - z} \cdot \frac{1}{2 - e^z} \sim \frac{1}{2} \cdot \frac{1}{\ln 2 - z} = \frac{1}{2\ln 2} \cdot \frac{1}{1 - (1/\ln 2)z} = \frac{1}{2\ln 2} \sum_{n=0}^{\infty} \left(\frac{1}{\ln 2}\right)^n z^n.$$

We now look for the next singularites (ordered by their distance from the origin). These are $\ln 2 \pm 2\pi i$. By L'Hôpital

$$\lim_{z \to \ln 2 \pm 2\pi i} \frac{z - \ln 2 \mp 2\pi i}{2 - e^z} = -\frac{1}{2}$$

und therefore

$$S(z) \sim -\frac{1}{2}\frac{1}{z - \ln 2 \mp 2\pi i} = \frac{1}{2\ln 2 \mp 2\pi i}\frac{1}{1 - \frac{z}{\ln 2 \mp 2\pi i}}$$

for $z \to \ln 2 \pm 2\pi i$. We can now use the Theorem of the lecture to determine the coefficients:

$$
\begin{aligned}
[z^n]S(z) &= \frac{1}{2}\left((\frac{1}{\ln 2})^{n+1} + (\frac{1}{\ln 2 + 2\pi i})^{n+1} + (\frac{1}{\ln 2 - 2\pi i})^{n+1}\right) + O(r)^{-n} \\
&= \frac{1}{2}\left((\frac{1}{\ln 2})^{n+1} + r^{n+1}(e^{i\phi(n+1)} + e^{-i\phi(n+1)})\right) + O(r)^{-n} \\
&= \frac{1}{2}\left((\frac{1}{\ln 2})^{n+1} + r^{n+1}\cos(\phi(n+1))\right) + O(r)^{-n}
\end{aligned}
$$

with $r = 1/\sqrt{\ln^2 2 + 4\pi^2} \approx 12.58547409739904$, $\phi = \arctan(\frac{2\pi}{\ln 2})$.

Homework Assignment 11-1 (10 points)

Determine the exponential growth of $[z^n]G(z)$, where

a) $G(z) = z^2/(1 - z - z^2)$,

b) $G(z) = \sqrt{1 + 2z} - \sqrt{2 + 2z - 4z^2}/\sqrt{3}$,

c) $G(z) = \ln(1 + \sin(z))/\ln(1 + \cos(z))$.

Homework Assignment 11-2 (10 points)

$$A(z) = \frac{\sqrt{1 - z^7}}{2z^2 - 3z + 1} \qquad B(z) = \frac{1 - z^2}{e^{z+3z^2}} \qquad C(z) = z^5 + 3z^2(z^3 + z^2 + 8)$$

Sort the sequences a_n, b_n, and c_n by their exponential growth.

Homework Assignment 11-3 (10 Points)

Determine $g_n = [z^n]G(z)$ up to an additive error of $O(4^n)$, where

$$G(z) = \sum_{n=0}^{\infty} g_n z^n = \frac{15z^2 + 8z + 1}{15z^2 - 8z + 1}.$$