Analysis of Algorithms

This tutorial is geared towards solving recurrence relations using generating functions. Here is an observation: If a_n satisfies the recurrence

$$a_n = x_1 a_{n-1} + x_2 a_{n-2} + \cdots + x_t a_{n-t}$$

for $n \geq t$, then the generating function $a(z) = \sum_{n\geq 0} a_n z^n$ is a rational function $a(z) = f(z)/g(z)$, where the denominator polynomial $g(z) = 1 - x_1 z - x_2 z^2 - \cdots - x_t z^t$ and the numerator polynomial is determined by the intial values $a_0, a_1, \ldots, a_{t-1}$ and has degree at most $t-1$. In fact, $f(z)$ may be written as:

$$f(z) = g(z) \sum_{n=0}^{t-1} a_n z^n \left(\bmod z^t\right).$$

Problem 9-1

Solve the following recurrence relations:

1. $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n > 2$ with $a_0 = 0$ and $a_1 = a_2 = 1$.
2. $a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3}$ for $n > 2$ with $a_0 = 0$, $a_1 = 1$, and $a_2 = 4$.

Solution

1. First write down $g(z)$ as $1 - 2z - z^2 + 2z^3$. Using the initial conditions, $f(z)$ may be written as:

$$\begin{aligned} f(z) &= g(z) \sum_{n=0}^{2} a_n z^n \pmod{z^3} \\ &= (z + z^2)(1 - 2z - z^2 + 2z^3) \pmod{z^3} \\ &= z - z^2 = z(1-z). \end{aligned}$$

Therefore,

$$a(z) = \frac{z(1-z)}{1 - 2z - z^2 + 2z^3} = \frac{z(1-z)}{(1-z)(1+z)(1-2z)} = \frac{z}{(1+z)(1-2z)}.$$

Using partial fractions, the last expression on the right hand side works out to:

$$\frac{z}{(1+z)(1-2z)} = \frac{1}{3}\left(\frac{1}{1-2z} - \frac{1}{1+z}\right),$$

from which we can read off $a_n = \frac{1}{3}(2^n - (-1)^n)$.

2. We start by writing down $g(z)$ as $1-5z+8z^2-4z^3$. Now using the initial conditions, $f(z)$ can be written as:

$$\begin{aligned} f(z) &= g(z)\sum_{n=0}^{2} a_n z^n = g(z)(z+4z^2) \pmod{z^3} \\ &= (1-5z+8z^2+4z^3)(z+4z^2) \pmod{z^3} \\ &= z(1-z). \end{aligned}$$

   Now $1-5z+8z^2-4z^3$ can be factorized as $(1-z)(1-2z)^2$, so that $a(z) = z/(1-2z)^2$. We know that

$$\frac{2z}{(1-2z)^2} = \sum_{n=0}^{\infty} n(2z)^n,$$

   and so $a_n = n2^{n-1}$ for $n \geq 3$. Actually, substituting $n = 0, 1, 2$, we see that this holds for $n \geq 0$.

Problem 9-2

In this problem, we will solve the *Quicksort* recurrence using OGFs. Typically when the coefficients of the recurrence are polynomials in the index n, then the relationship constraining the generating function is a differential equation. Recall that the Quicksort recurrence is:

$$nC_n = n(n+1) + 2\sum_{k=1}^{n} C_{k-1}, \text{ for } n \geq 1 \text{ with } C_0 = 0.$$

Define $C(z) = \sum_{n\geq 0} C_n z^n$.

1. Multiply both sides of the recurrence by z^n and sum over the index n to obtain a functional relation involving $C(z)$, $C'(z)$, and z of the form:

$$C'(z) + P(z)C(z) = Q(z),$$

   where $P(z)$ and $Q(z)$ are functions of z.

2. This differential equation can be solved by multiplying both sides by the "integrating factor" $e^{\int_0^z P(x)\mathrm{d}x}$. Notice that multiplying by this factor yields:

$$\begin{aligned} C'(z)e^{\int_0^z P(x)\mathrm{d}x} + P(z)C(z)e^{\int_0^z P(x)\mathrm{d}x} &= Q(z)e^{\int_0^z P(x)\mathrm{d}x} \\ \left(C(z)e^{\int_0^z P(x)\mathrm{d}x}\right)' &= Q(z)e^{\int_0^z P(x)\mathrm{d}x} \end{aligned}$$

   We can now express $C(z)$ as

$$C(z) = e^{-\int_0^z P(x)\mathrm{d}x} \int Q(z)e^{\int_0^z P(x)\mathrm{d}x}\mathrm{d}z.$$

Solution

As suggested, multiplying both sides of the recurrence by z^n and then summing over the index n yields

$$\sum_{n=0}^{\infty} nC_n z^n = \sum_{n=0}^{\infty} n(n+1)z^n + 2\sum_{n=0}^{\infty}\left(\sum_{k=1}^{n} C_{k-1}\right) z^n. \tag{1}$$

Now $\sum_{n\geq 0} nC_n z^n = zC'(z)$. Note that

$$\begin{aligned}\sum_{n=0}^{\infty} n(n+1)z^n &= 2\sum_{n=0}^{\infty}\binom{n+1}{2}z^n\\ &= \frac{2}{z}\sum_{n=0}^{\infty}\binom{n+1}{2}z^{n+1}\\ &= \frac{2}{z}\sum_{n=2}^{\infty}\binom{n}{2}z^n\\ &= \frac{2}{z}\frac{z^2}{(1-z)^3}\\ &= \frac{2z}{(1-z)^3}.\end{aligned}$$

Finally, note that

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\sum_{k=1}^{n} C_{k-1}\right) z^n &= \sum_{n=0}^{\infty} z^n \sum_{n=0}^{\infty} C_n z^{n+1}\\ &= \frac{zC(z)}{1-z}.\end{aligned}$$

Hence the functional relation that we are seeking is:

$$C'(z) - \frac{2}{1-z}C(z) = \frac{2}{(1-z)^3}.$$

The integrating factor in our case is

$$e^{-2\int_{x=0}^{z}\frac{1}{1-z}\mathrm{d}x} = e^{2\ln(1-z)} = (1-z)^2.$$

Hence $C(z)$ is given by

$$C(z) = \frac{1}{(1-z)^2}\int \frac{2}{1-z}\mathrm{d}z = \frac{2}{(1-z)^2}\ln\frac{1}{1-z}.$$

Using the fact that $\frac{z}{(1-z)^2}\ln\frac{1}{1-z} = \sum_{n\geq 0} n(H_n - 1)z^n$, we obtain:

$$[z^n]\frac{2}{(1-z)^2}\ln\frac{1}{1-z} = 2(n+1)(H_{n+1}-1).$$

Homework Assignment 9-1 (10 Points)
Solve the recurrences:

1. $a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3}$ for $n > 2$ with $a_0 = 1$, $a_1 = 2$, and $a_2 = 4$.
2. $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$ for $n > 2$ with $a_0 = a_1 = 0$ and $a_2 = 1$.

Hint: Use might need to use:

$$\frac{1}{(1-z)^{m+1}} = \sum_{n=0}^{\infty} \binom{n+m}{m} z^n.$$

Homework Assignment 9-2 (10 points)
Use generating functions to solve the recurrence:

$$na_n = (n-2)a_{n-1} + 2 \quad \text{for } n > 2 \quad \text{with } a_1 = 1, a_2 = 1.$$