

Analysis of Algorithms

Problem 8-1

Find the ordinary generating function (OGF) for each of the following sequences:

- (1) $\{k2^{k+1}\}_{k \geq 0}$ (2) $s_0 = 0$ and $s_k = \frac{1}{k}$ for $k \geq 1$ (3) $\{H_k\}_{k \geq 1}$
(4) $\{kH_k\}_{k \geq 1}$ (5) $\{k^3\}_{k \geq 2}$

Solution

1. The ordinary generating function for the sequence $\{k2^{k+1}\}_{k \geq 0}$ is:

$$\sum_{k \geq 0} k2^{k+1}z^k = \sum_{k \geq 0} 2k(2z)^k = 2z \sum_{k \geq 1} 2k(2z)^{k-1} = 2z \left(\sum_{k \geq 0} (2z)^k \right)'.$$

Now the last term may be rewritten as:

$$2z \left(\frac{1}{1-2z} \right)' = \frac{4z}{(1-2z)^2}.$$

2. Start with the generating function $G(z) = \sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$. Integrating both sides we obtain:

$$\begin{aligned} \int_0^z \sum_{k=0}^{\infty} z^k dz &= \int_0^z \frac{1}{1-z} dz \\ &= -\ln(1-z) \Big|_0^z \\ &= \ln \frac{1}{1-z}. \end{aligned}$$

The series itself can be integrated term-by-term. This is valid if z is in the radius of convergence of the power series, and we can make this assumption because the value of z itself is not of interest to us. This then gives us:

$$\ln \frac{1}{1-z} = \sum_{k=0}^{\infty} \frac{z^{k+1}}{k+1} = \sum_{k=1}^{\infty} \frac{z^k}{k}.$$

Since $s_0 = 0$, the OGF of the sequence is $\ln(1-z)^{-1}$.

3. Let s_k be as defined before: $s_0 = 0$ and $s_k = 1/k$ for $k \geq 1$. Also define $H_0 = 0$. Then the OGF for $\{H_k\}_{k \geq 0}$ is

$$\begin{aligned} \sum_{k=0}^{\infty} H_k z^k &= \sum_{k=0}^{\infty} \left(H_0 + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} \right) z^k \\ &= \sum_{k=0}^{\infty} \left(\sum_{i=0}^k 1 \cdot s_{k-i} \right) z^k \\ &= \sum_{k=0}^{\infty} z^k \sum_{k=0}^{\infty} s_k z^k \\ &= \frac{1}{1-z} \ln \frac{1}{1-z}. \end{aligned}$$

4. We start with the identity obtained in the last exercise: $\sum_{k \geq 0} H_k z^k = \frac{1}{1-z} \ln \frac{1}{1-z}$. Differentiating both sides and multiplying by z , we obtain:

$$\sum_{k=1}^{\infty} k H_k z^k = \frac{z}{(1-z)^2} \left(1 + \ln \frac{1}{1-z} \right).$$

Now the index k in power series on the left hand side may be written so that it ranges from 0 to ∞ . This shows that the expression on the left hand side is the closed-form of the generating function that we are seeking.

5. We start with the fact that $\sum_{k \geq 0} z^k = \frac{1}{1-z}$. Differentiating both sides and using the fact that a power series can be termwise differentiated w.r.t z if z is in the radius of convergence, we obtain:

$$\sum_{k=1}^{\infty} k z^{k-1} = \ln \frac{1}{1-z}. \quad (1)$$

Multiplying both sides by z and differentiating again, we obtain:

$$\sum_{k=1}^{\infty} k^2 z^{k-1} = \ln \frac{1}{1-z} + \frac{z}{1-z}. \quad (2)$$

We again multiply both sides by z and differentiate, to obtain:

$$\sum_{k=1}^{\infty} k^3 z^{k-1} = \ln \frac{1}{1-z} + \frac{3z}{1-z} + \frac{z^2}{(1-z)^2}. \quad (3)$$

Again multiplying the above by z , we obtain:

$$\sum_{k=1}^{\infty} k^3 z^k = z \ln \frac{1}{1-z} + \frac{3z^2}{1-z} + \frac{z^3}{(1-z)^2}. \quad (4)$$

Now the generating function for our sequence is $\sum_{k \geq 2} k^3 z^k$ and this equals the right hand side of (4) minus z , that is:

$$z \ln \frac{1}{1-z} + \frac{3z^2}{1-z} + \frac{z^3}{(1-z)^2} - z.$$

Problem 8-2

```
1 func( int n ){  
    int s = 0;  
3    if ( n == 0 ) return 1;  
    for( int i = 0; i < n; ++i )  
5        s += func( i );  
    return s;  
7 }
```

Compute how often the 5th line of this program is executed using (ordinary) generating functions.

Solution

Let A_n denote the how often the 5th line is called on input n . We immediately obtain $A_0 = 0$ and $A_n = n + \sum_{i=0}^{n-1} A_i$. The corresponding generating function is thus

$$A(z) := \sum_{n=0}^{\infty} A_n z^n = \sum_{n=0}^{\infty} n z^n + \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n-1} A_k \right) z^n.$$

Now the last term on the right hand side may be seen as the convolution of the series $\sum_{n=0}^{\infty} z^n$ and $\sum_{n=0}^{\infty} A_n z^{n+1}$. The convolution product is simply $zA(z)/(1-z)$ so that we may write:

$$A(z) = \frac{z}{(1-z)^2} + \frac{zA(z)}{1-z},$$

which implies:

$$A(z)(1-2z) = \frac{z}{1-z}$$

and therefore

$$A(z) = \frac{1}{(1-z)(1-2z)} z.$$

Let $\frac{1}{(1-z)(1-2z)} = \frac{a}{1-z} + \frac{b}{1-2z}$ for some a, b . Setting $z = 0$ implies $a + b = 1$ and $z = -1$ implies $a/2 + b/3 = 1/6$. We easily obtain $a = -1$ and $b = 2$ and therefore

$$A(z) = \left(-\frac{1}{1-z} + \frac{2}{1-2z} \right) z = -\sum_{n=0}^{\infty} z^{n+1} + \sum_{n=0}^{\infty} (2z)^{n+1} = \sum_{n=1}^{\infty} (2^n - 1) z^n.$$

Thus $A_0 = 0$ and $A_n = 2^n - 1$ for all $n \geq 1$.

Homework Assignment 8-1 (10 Points)

Solve this recurrence using generating functions:

$$a_n = 2a_{n-1} + 3a_{n-2}$$

and $a_0 = 0, a_1 = 2$.

Homework Assignment 8-2 (10 points)

Find $[z^n]$ for each of the following OGFs.

$$\frac{1}{(1-3z)^4}, \quad (1-z)^2 \ln \frac{1}{1-z}, \quad \frac{1}{(1-2z^2)^2}$$