**RWTH Aachen**
**Lehrgebiet Theoretische Informatik**
Reidl—Rossmanith—Sanchez—Sikdar

**SS 2014**
Final Exam
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# Analysis of Algorithms

This exam is of two hours duration. Please write down your full name and matriculation number on each sheet of paper that you write on. Please hand in all sheets including the ones you have used for rough work. Write legibly and show all your steps.

**Exercise 1** (10 Points)
Consider the following algorithm for searching an array $a[1, \ldots, n]$ for an element $x$. We assume that the array is sorted in increasing order and that the element $x$ is at some random location in the array. Let $B_n$ be the expected number of comparisons on an $n$-element array. Write down a recurrence for $B_n$. What is $B_3$?

**Algorithm: Binary Search with randomly chosen pivot element**

1. Choose randomly and with uniform probability an $i \in \{1, \ldots, n\}$.
2. If $a[i] = x$, output $i$ and halt.
3. Continue recursively on the left subarray, if $x < a[i]$, or the right subarray, if $x > a[i]$.

**Solution**
There are two cases to consider here: The first is that the element $x$ is found at the randomly chosen location $i$. This happens with a probability of $1/n$. With a probability of $1 - 1/n$, the search continues and the element is found at the recursive step. Now if the element $x$ is found at the recursive step then the expected number of comparisons made is:

$$1 + \frac{1}{n}\left(\sum_{k=1}^{n} \frac{k-1}{n-1} B_{k-1} + \sum_{k=1}^{n} \frac{n-k}{n-1} B_{n-k}\right).$$

This may be explained as follows: In this case, one comparison is made and the search is carried on in either the left or right subarray. Now the probability that the index chosen is $k$ is $1/n$. The probability that the element being searched for is in the left subarray is $(k-1)/(n-1)$, since there are $n-1$ possibilities and there are $k-1$ of them to the left. The last term above is the expected number of comparisons made if the element is in the right subarray. Now the expected number of comparions is:

$$B_n = \frac{1}{n} + \frac{n-1}{n}\left(1 + \sum_{k=1}^{n}\left(\frac{k-1}{n-1} B_{k-1} + \frac{n-k}{n-1} B_{n-k}\right)\right).$$

This may be written as follows:

$$B_n = 1 + \frac{2}{n^2} \sum_{k=0}^{n-1} k B_k.$$

Now, $B_1 = 1$, $B_2 = \frac{3}{2}$, and $B_3 = 17/9$.

**Exercise 2** (10 Points)
Suppose that poker chips come in four colors—red, white, green, and blue. Find and solve a recurrence relation for the number of ways to stack $n$ poker chips such that there are no two consecutive blue chips.

**Solution**
Let the number of ways of stacking $n$ poker chips such that no two consecutive chips are blue be $A_n$. If the first chip is either red, white or green there are $A_{n-1}$ ways (in each case) of stacking the remaining $n-1$ chips. If the first chip is a blue, then the second chip must be either red, green, or white and there are $A_{n-2}$ ways (in each case) of stacking the remaining $n-2$ chips. We have $A_0 = 1$ (the empty stack satisfies the given conditions) and $A_1 = 4$. Thus the recurrence we are seeking is:

$$A_n = 3A_{n-1} + 3A_{n-2}.$$

The solution works out to:

$$A_n = \frac{5+\sqrt{21}}{2\sqrt{21}}\left(\frac{3+\sqrt{21}}{2}\right)^n - \frac{5-\sqrt{21}}{2\sqrt{21}}\left(\frac{3-\sqrt{21}}{2}\right)^n.$$

**Exercise 3** (10 Points)
Find an expression for

$$[z^n]\frac{1}{\sqrt{1-z}}\ln\frac{1}{1-z}.$$

Your solution can include a sum!

**Solution**
Looking up our table of OGFs, we see that $\ln\frac{1}{1-z}$ is the OFG for the sequence

$$0, 1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots$$

Define $s_0 = 0$ and $s_n = \frac{1}{n}$ for all $n \geq 1$. We may now write down the given function as:

$$\begin{aligned}\frac{1}{\sqrt{1-z}}\ln\frac{1}{1-z} &= \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-z)^n \cdot \sum_{n=0}^{\infty} s_n z^n \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{-\frac{1}{2}}{k}(-1)^k s_{n-k}\right) z^n.\end{aligned}$$

Now the term inside the inner sum evaluates to

$$\sum_{k=0}^{n-1}\frac{1}{4^k\cdot(n-k)}\binom{2k}{k}.$$

**Exercise 4** (10 Points)
Sort the series with the following generating functions by their asymptotic growth. Justify your steps!

1. $A(z) = \frac{1}{\sqrt{1-\frac{z^2}{2}}}$.
2. $B(z) = \frac{z}{2-3z+z^2}$.
3. $C(z) = e^{z+z^2}$.

**Solution**
The function $A(z)$ has a singularity at $z = \sqrt{2}$ and hence $[z^n]A(z)$ grows as $1/2^{n/2}$. The function $B(z)$ has a singularity at $z \in \{1, 2\}$. Since $z = 1$ is the singularity nearest to the origin, $[z^n]B(z)$ grows as $1^n$. In fact, if we expand the terms, we see that $[z^n]B(z) = 1-\frac{1}{2^n}$. The function $C(z)$ does not have any singularities. Observe that $C(z)$ is analytic over the entire complex plane. Using the saddle-point bounds, we obtain that

$$[z^n]C(z) \leq \min_r \max_{|z|=r} \frac{|C(z)|}{r^n}.$$

Moreover, the generating function $C(z)$ has non-negative coefficients, and hence the maximum is attained on the real line and we obtain:

$$[z^n]C(z) \leq \min_r \frac{C(r)}{r^n}.$$

The minimum value may be obtained by first setting $h(r) = C(r)/r^n$ and differentiating. Omitting details, the minimum occurs at $r = \sqrt{n/2}$. Thus

$$[z^n]C(z) = \frac{e^{\sqrt{n/2}+n/2}}{(n/2)^{n/2}}.$$

Now the asymptotic growth of the coefficients of three functions is clear: $[z^n]C(z)$ is the slowest growing function, followed by $[z^n]A(z)$, followed by $[z^n]B(z)$.